II

FUNDAMENTAL POINTS OF POTENTIAL THEORY

This paper is a study of the Stieltjes integral in connection with potential theory. The Stieltjes integral is well adapted to the investigation of problems of mathematical physics, first because it applies equally well to discrete and continuous sequences of values, and thus enables either to be regarded as an approximation to the other, and in the second place because it is based on additive functions of point sets, or in special cases additive functions of points, curves, and surfaces, of limited variation. These latter are familiar to us in volume, point, curvilinear and surface distributions of mass and electricity.

Laplace's equation also remains fundamental. Even if complete statements require a four dimensional space, whenever a three dimensional space is used Laplace's equation will be merely the statement that one is dealing with a conservative field of force, without divergence. And the general vector field in three dimensions can be built up out of two special vector fields, one being of this precise character, and indeed both of them bearing a special relation to Laplace's equation.

Part 1 of the paper develops the general properties of the potential function expressed as a Stieltjes integral, and demonstrates the relation of the potential function thus defined to the integral form of *Poisson's* equation, which applies to any sort of distribution of mass. In particular it is shown that the differential form of *Laplace's* equation is not less general than the integral form. Part II is concerned with general forms of Green's theorem as applied to polarization vectors and solutions of *Poisson's* equation. Part III is devoted to the boundary value problem for harmonic functions and the general open region.

These studies originated in 1907, when it first became apparent to me that the theory was unnecessarily complicated by the form of the Laplacian operator, but I did no work on the subject until 1913, when it occurred to me to use instead of the operator

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) u,$$

the operator

$$\lim_{h=0} \frac{u(x+h,\, y)+u(x,\, y+h)+u(x-h,\, y)+u(x,\, y-h)-4\, u(x,\, y)}{h^2},$$

or the operator

$$\lim_{\sigma=0} \int_s \frac{\partial u}{\partial n}\, d s,$$

where s is a closed contour containing an area σ which is allowed to approach zero. The first of these operators had been used by H. Petrini, and conditions for the existence of the limit discussed.[1]) The second idea, under the form of the equation

$$\int_s \frac{\partial u}{\partial n}\, d s=0,$$

had, it turned out, been discussed by Bôcher,[2]) with relation

[1]) H. Petrini, "Les derivées premiers et secondes du potentiel," Acta Mathematica, vol. 31 (1908), pp. 127–332. Hereafter references to an article already cited will be made by quoting the number of the reference.

[2]) M. Bôcher, "On harmonic functions in two dimensions," Proceedings of the American Academy of Science, vol. 41 (1905–06).

to Laplace's equation; of the two it is obviously the concept which is the more closely allied with the physical interpretation. The fundamental problem from this point of view was discussed by C. W. Oseen.[3])

In spite of the generality and importance of Oseen's discussion I have spoken of these general equations, as Integro-differential equations of Bôcher type, since apparently it was Bôcher who first considered them from this point of view. The operator has been considered somewhat roughly by Ignatowsky, where in three dimensions it occurs as a vector operator.[4]) Further references and a systematic treatment from the point of view of Riemann integration of the equations of elliptic, hyperbolic and parobolic types will be found in the Cambridge Colloquium.[5])

The present paper is an endeavor to obtain results commensurate in scope with the general theory of integration, thus to continue in the direction pointed out by C. de la Vallée-Poussin in his treatment of additive functions of point sets.[6]) The author in this connection takes the opportunity of acknowledging how much he has benefited by the exchange of ideas in conversation with his colleagues, especially with P. J. Daniell, whose studies on a general form of integral are now available.[7]) The theory has been worked out here for two dimensions only, but much of the material is obviously independent of the number of dimensions. For

[3]) C. W. Oseen, "Uber die Bedeutung der Integralgleichungen in der Theorie der Bewegung einer reibenden, unzusammendrückbaren Flüssigkeit," Arkiv för Mathematik, Astronomi och Fysik, Band 6 (1911), No. 23.

[4]) Ignatowsky, "Die Vektoranalysis," Leipzig (1909–10).

[5]) Evans, "Functionals and Their Applications," The Cambridge Colloquium (1916), Part I, New York, 1918, Lecture IV.

[6]) C. de la Vallée-Poussin, "Intégrales de Lebesgue," Paris (1916).

[7]) P. J. Daniell, "A general form of integral," Annals of Mathematics, Vol. XIX, No. 4 (1918), and continued in later papers.

the working out of the rest, the author is counting on the help of his colleague, H. E. Bray, who has already published a study of Green's theorem in terms of Lebesgue integrals.[8])

The central equations under discussion are the following:

$$\int_s \nabla_n u\, ds = 0, \tag{1}$$

$$\int_s \nabla_n u\, ds = F(s), \tag{2}$$

where $F(s)$ is an additive function of closed curves s, and $\nabla_n u$ represents the normal component of the gradient of u.

1. *Stieltjes Potentials*

1. Stieltjes integrals. Additive functions of point sets. Let Σ be a bounded domain in the plane, and $f(e)$ an additive function of point sets, defined for sets e of Σ measurable in the sense of Borel. Such a function can be written as the difference of two additive functions each of which never becomes negative for any set e, one of them corresponding to the positive and the other to the negative variation of $f(e)$:

$$f(e) = p(e) - n(e).$$

The function of point sets

$$t(e) = p(e) + n(e),$$

[8]) H. E. Bray, "A. Green's theorem in terms of Lebesgue integrals," Annals of Mathematics, Vol. 21 (1920), pp. 141–156.

A form of Green's theorem has been proved by P. J. Daniell, for a general region, but with more restricted functions; see

[9]) P. J. Daniell, "A general form of Green's theorem," Bulletin of the American Mathematical Society, Vol. 25 (1919), pp. 353–357.

A first paper on the present theory has been published in the article:

[10]) G. C. Evans, "Sopra un'equazione integro-differenziale di tipo Bôcher," Rend. della R. Accad. dei Lincei, Vol. 28, 1° sem. (1919), pp. 262–265.

and a summary of the present paper will appear in the article:

[11]) G. C. Evans, "Problems of potential theory," Proceedings of the Nat. Academy of Sciences, Vol. 7, 1921.

which is additive and also not negative, is the total variation function of $f(e)$.*

We might understand $f(e)$ physically, by assuming that an electric charge is somehow distributed on points, curves and areas in the region Σ, and letting $f(e)$ represent the total charge on the set of points e.

In contradistinction with functions of point sets we consider also of course functions of x, y in the usual sense. Such functions are point functions, and for brevity we shall write them as functions of a single argument M, the point whose coördinates are x, y.

1.1. For integrands $q(M)$ which are continuous, the definition of the Stieltjes integral.

$$\int q(M)\, d\, f(e), \tag{3}$$

is well known in terms of a Riemann summation. By making a division of Σ in mutually exclusive domains e_i of diameter $\leq \delta$, and taking M_i a point of e_i, we arrive at the customary definition

$$\int_\Sigma q(M)\, d\, f(e) = \lim_{\delta=0} \Sigma_i\, q(M_i) f(e_i), \tag{4}$$

and consequently can justify the equation

$$\int_\Sigma q(M)\, d\, f(e) = \int_\Sigma q(M)\, d\, p(e) - \int_\Sigma q(M)\, d\, n(e). \tag{5}$$

In the case of $q(M) = \text{const.}$, equation (4) gives the result

$$\int_\Sigma c\, d f(e) = c\, f(\Sigma). \tag{5'}$$

In general, if $q(M)$ is continuous, it is at once obvious that

$$\left| \int_\Sigma q(M)\, d\, f(e) - \Sigma_i\, q(M_i)\, d f(e_i) \right| \leq \omega_\delta\, t(\Sigma), \tag{5''}$$

where ω_δ is the upper limit of the oscillation of $q(M)$ in a point set of diameter $\leq \delta$.

* See [6]), p. 58.

To integrate $q(M)$ with respect to $f(e)$ over a set E in Σ, measurable in the sense of Borel, it is sufficient to define a new additive function of point sets $\phi(e)$

$$\phi(e) = f(Ee),$$

and define

$$\int_E q(M)\, df(e) = \int_\Sigma q(M)\, d\phi(e). \tag{4'}$$

Properties analogous to those expressed by equations (5), (5′) and (5″) are seen to hold.

1.2. In the main, we shall use functions $q(M)$ which are continuous, and certain others which are the limits of continuous functions. Nevertheless it is desirable to have at hand the principal theorems for the more general case, and these are accordingly reproduced below.*

If the function $q(M)$ is not continuous, the integral is defined by limiting and linear processes, and the equations expressed by these processes are maintained as properties of the integral for all functions which are measurable in the Borel sense, and have integrals. The equation (5) is such a property, also the following ones.†

(6′) If $q_j(M)$ constitute an increasing sequence of functions, such that $\int q_j(M)\, dt(e)$ exists for all values $j = 1, 2, 3, \ldots$, and has a limit as n becomes infinite, then

$$\lim_{j=\infty} \int q_j(M)\, df(e) = \int \lim_{j=\infty} q_j(M)\, df(e).$$

(6″) If $\lim\limits_{j=\infty} q_j(M) = q(M)$, where $\int q_j(M)\, dt(e)$ exists for all

*For those properties which are essential merely for the study of Stieltjes potentials, see [10]. For a more general and systematic study, see [7].

† A different method of extending the field of definition of the integral is suggested by Lebesgue.

[12]) H. Lebesgue, "Sur l'intégrale de Stieltjes et sur les operations fonctionelles lineaires," Comptes Rendus h. de l'Acad. des Sciences, Vol. 150 (1910) pp. 86–88.

values $j=1, 2, \ldots$ and $|q_j(M)| < \bar{q}(M)$ where $\int \bar{q}(M)\, dt(e)$ exists, then $\int q(M)\, df(e)$ exists and

$$\lim_{j=\infty} \int q_j(M)\, df(e) = \int q(M)\, df(e).$$

(6‴) If $q(M) = a\, q_1(M) + b\, q_2(M)$ where $\int q_1(M)\, dt(e)$ and $\int q_2(M)\, dt(e)$ exist, then

$$\int q(M)\, df(e) = a\int q_1(M)\, df(e) + b\int q_2(M)\, df(e).$$

In particular, the property (6′) provides a definition for the integral for functions $q(M)$ that become infinite at certain points M.

1.3. It will be necessary to consider iterated integrals, and functions $q(M_1, M)$ depending on two point arguments. If $q(M_1, M)$ is continuous in both point arguments we have such identities as the following: [13])

$$(7) \quad \int_\Sigma d\, g(e_1) \int_\Sigma q(M_1, M)\, df(e) = \int_\Sigma df(e) \int_\Sigma q(M_1, M)\, d\, g(e_1)$$

$$(7') \quad \int_\sigma d\, \sigma_1 \int_\Sigma q(M_1, M)\, df(e) = \int_\Sigma df(e) \int_{\sigma_1} q(M_1, M)\, d\, \sigma_1.$$

In fact, these identities follow immediately from the approximation formula (5″).

By means of (6′) and (6‴) the extension to bounded functions measurable in the Borel sense is also immediate. If the function $q(M_1, M)$ is not bounded we may write

$$q_n(M_1, M) = |\, q(M_1, M)\,| \text{, if } |\, q(M_1, M)\,| \leq n.$$
$$= n \qquad\qquad \text{, otherwise,}$$

and in order for (7) to hold it is sufficient that

$$\int_\Sigma dt_g(e) \int_\Sigma q_n(M_1, M)\, dt_f(e),$$

[13]) W. H. Young, "Integration with respect to a function of bounded variation," Proceedings London Mathematical Society, Vol. 13 (1914), pp. 97–150; see page 149.

where t_f and t_g denote the total variation functions of $f(e)$ and $g(e)$ respectively, remain finite; for (7′) to hold it is sufficient that

$$\int_\Sigma d\,t_f(e) \int_{\sigma_1} q_n(M_1, M)\; d\,\sigma_1 = \int_{\sigma_1} d\,\sigma_1 \int q_n\,(M_1, M)\; d\,t_f(e)$$

remains finite as n becomes infinite. In fact, since the right-hand member remains finite, as n becomes infinite, it follows that $\lim\limits_{n=\infty} \int_\Sigma q_n(M_1, M)\; d\,t_f(e)$ exists except possibly for points M_1, which form a point set of superficial measure zero. Hence, by (6), $\int_\Sigma q(M_1, M)\; d\,f(e)$ exists everywhere in Σ except possibly at points M_1, which form a set of superficial measure zero.

In particular, the identity (7′) is valid if

$$|\,q(M_1, M)\,| \leq \frac{\text{const.}}{r},$$

where r denotes the distance $M_1\,M$. For in this case,

$$\int_{\sigma_1} q_n(M_1, M)\; d\,\sigma_1 \leq \pi\; d \;\text{const.}$$

$$\int_\Sigma d\,t_f(e) \int_{\sigma_1} q_n(M_1, M)\; d\,\sigma_1 \leq \pi\; t_f(\Sigma)\; d\;\text{const.},$$

where d is the diameter of σ.

As we progress, we shall need to obtain the corresponding equations for certain particular cases, where the integral of the Stieltjes integral is taken over a curve instead of a surface. Here however let it suffice to consider a single case, where the integration is extended along a straight line.

1.4. Let q involve a parameter α as well as M, and be *absolutely continuous* as a function of α, for every M. Let $q'(M, \alpha) = \partial\, q/\partial\, \alpha$ be *measurable in the Borel sense* as a function of x, y, α and let

$$\Phi(\alpha) = \int_\Sigma q(M, \alpha)\; d\,f(e).$$

If for $q'(M, \alpha)$ we define the function $q_n'(M, \alpha)$, analogous to the function q_n above, and if

$$\int_\Sigma d\,t(e)\int_{\alpha_0}^{\alpha_1} q'_n(M, \alpha)\, d\,\alpha,$$

remains finite as n becomes infinite, we may write

$$\int_{\alpha_0}^{\alpha} d\,\alpha \int_\Sigma q'(M, \alpha)\, d\,f(e) = \int_\Sigma d\,f(e) \int_{\alpha_0}^{\alpha} q'(M, \alpha)\, d\,\alpha,$$
$$\alpha_0 \leq \alpha \leq \alpha_1,$$
$$= \Phi(\alpha) - \Phi(\alpha_0),$$

and therefore *the function*

$$\Phi'(\alpha) = \int_\Sigma q'(M, \alpha)\, d\,f(e)$$

is the derivative of $\Phi(\alpha)$

$$\Phi'(\alpha) = \frac{d}{d\,\alpha}\Phi(\alpha) \tag{8}$$

wherever $\Phi'(\alpha)$ *is the derivative of its indefinite integral*, that is, except for values of α which constitute at most a set of zero measure.

2. Stieltjes integrals. Additive functions of curves. The integrals in the foregoing paragraphs have been based on additive functions of point sets. In one dimension, an additive function of point sets is the variation of a function of a single variable, of limited variation. In two dimensions, functions of curves may be considered as corresponding to functions of one variable in one dimension, from this point of view; and we shall find it convenient to make use, therefore, of *additive functions of curves of limited variation*, and to construct Stieltjes integrals with respect to them.

The curves upon which such functions are defined might be discussed from two points of view. On the one hand we might seek to limit ourselves to the simplest class of curves, for example circles or rectangles, and obtain all our results by means of them. On the other hand, we might try to extend our results to the most general classes of curves. If we should know how the discontinuities of an additive function of curves were distributed, it would be sufficient otherwise merely to know its value for rectangles in order to determine its value throughout. But the necessity of knowing in advance the distribution of discontinuities makes the problem different from that of additive functions of point sets, where by means of the values of the function for sets related to rectangles the function is determined for every Borel-measurable set. Speaking roughly, in an additive function of curves there would be determined for any curve an outward and an inward value of the function but not a value on the curve itself. The situation is exemplified by the case of curvilinear integrals, where one closed curve may be the frontier of many different point sets.

2.1. We shall for the most part restrict ourselves to what we shall call *curves of class* Γ. A curve of class Γ is a simple closed rectifiable curve composed of a finite number of pieces, each piece having a definite direction at any point; for each curve there is a constant Γ such that the following inequality holds:

$$\int_s \frac{|\cos nr|}{r} ds < \Gamma$$

in which r represents the distance M_1M, M being the argument subject to the integration, and M_1 an arbitrary point. For a given curve the constant Γ is to be independent of the position of M. Thus for a curve which is everywhere convex, Γ may be taken as 2π.

Let s_1 and s_2 be two closed curves of Γ exterior to each other except for a common portion s', and let s_3 be the curve composed of s_1, and s_2 with the omission of s'; if now for all such curves, the relation

$$F(s_1)+F(s_2)=F(s_3)$$

is satisfied, the functional $F(s)$ is said to be an additive function of curves.

2.2. The additive function of curves $F(s)$ is said to be of *limited variation* in Σ if it may be written as the difference of two not-negative additive functions of curves

$$F(s)=P(s)-N(s)$$

the functionals $P(s)$ and $N(s)$ being bounded in value for all curves of class Γ in Σ.

The Stieltjes integral

$$\int_\sigma q(M)\, d\, F(s),$$

when $q(M)$ is continuous, may be defined by the same process as the integral with respect to $f(e)$. The region σ bounded by s is subdivided into mutually exclusive subregions bounded by curves s_i of Γ, no region being of diameter $>\delta$; then by definition

$$\int_\sigma q(M)\, d\, F(s)=\lim_{\delta=0}\Sigma_i\, q(M_i')\, F(s_i) \tag{9}$$

It follows then as in (5''), that

$$\left|\int_\sigma q(M)\, d\, F\,(s)-\Sigma_i\, q(M_i')\, F(s_i)\right|\leq \omega_\delta\, T(s)$$

where $T(s)=P(s)+N(s)$ is the total variation function of $F(s)$. All the properties (5) to (8) may in turn be developed for these new integrals, in precisely the same way as before.*

3. Certain particular integrals. We wish now to consider various integrals based upon the function $r=M_1M$. As a matter of notation, when we are to treat angles between

* Both of these are, of course, special cases of the general integral discussed by Daniell; see [7]).

the direction of r and other directions, we shall take as the sense of r the direction of M_1M.

From the discussion of Art. 1.4, it may be concluded that the integrals $\int_\Sigma \frac{\cos x\, r}{r}\, d\, f(e)$, $\int_\Sigma \log\frac{1}{r}\, d\, f(e)$ converge except for points M which form at most a set of superficial density zero. Further, the identity

$$(10)\qquad \int_{\sigma_1} d\,\sigma_1 \int_\Sigma \frac{\cos x\, r}{r}\, d\, f(e) = \int_\Sigma d\, f(e) \int_{\sigma_1} \frac{\cos x\, r}{r}\, d\,\sigma_1$$

is valid for any region σ_1. Equation (10) remains valid if for Σ we substitute an arbitrary σ bounded by a curve of class Γ and for $f(e)$ an $F(s)$, additive and of limited variation.

3.1. The integral $\int_\Sigma \frac{\cos n\, r}{r}\, d\, f(e)$ where n represents the normal (directed internally) at points M_1 of a curve s_1 of Γ, is itself summable over any curve of Γ, and, for any segment of a curve of Γ, exists except possibly at the points of a set of null linear measure. In fact, consider the function

$$q_m(M_1, M) = \frac{|\cos n\, r|}{r}, \text{ if } \frac{|\cos n\, r|}{r} < m$$
$$= m, \text{ otherwise.}$$

Then the quantity

$$\int_{\sigma_1} d\, s_1 \int_\Sigma q_m(M_1, M)\, d\, t(e) = \int_\Sigma d\, t(e) \int_{s_1} q_m(M_1, M)\, d\, s_1$$
$$\leq \Gamma\, t(\Sigma)$$

remains finite as m becomes infinite. Hence the equation

$$(11)\qquad \int_{s_1} d\, s_1 \int_\Sigma \frac{\cos n\, r}{r}\, d\, f(e) = \int_\Sigma d\, f(e) \int_{s_1} \frac{\cos n\, r}{r}\, d\, s_1$$

of which the right-hand member will eventually be evaluated, is verified. Here again, we may substitute σ for Σ and $F(s)$ for $f(e)$.

3.2. Finally we must consider the integral

$$\int_{\Sigma} \log \frac{1}{r}\, d\, f(e)$$

to be integrated as a function of M_1 along a closed curve of Γ with respect to an arbitrary direction y. As in the case of the function just treated, we have the identity

$$\int_{s_1} d\, y_1 \int_{\Sigma} \log \frac{1}{r}\, d\, f(e) = \int_{\Sigma} d\, f(e) \int_{s_1} \log \frac{1}{r}\, d\, y_1 \tag{12}$$

for any curve for which $\int_{s_1} \log \frac{1}{r} \mid d\, y_1 \mid$ exists, and is bounded with respect to the position of M, M being the argument subject to the integration. Such curves do not include all those of class Γ although the latter are themselves rectifiable.*

3.3. For more general curves the integral demands further investigation. By means of the identity

$$\int_{s_1} \log \frac{1}{r}\, d\, y_1 = \int_{\sigma_1} \frac{\cos x\, r}{r}\, d\, \sigma_1 \tag{12'}$$

which is valid when s_1 is any simple rectifiable curve for which $\int_{s_1} \log \frac{1}{r} \mid d\, y_1 \mid$ exists, and in particular for any rectangle, there is defined when M is fixed, a unique additive function of point sets for any set measurable in the Borel sense; it is moreover in this case absolutely continuous since it is given by a superficial integral.† It is thus a definite function of curves, having the same value for all the point sets which have a given curve as frontier.

As far then as we regard a curvilinear integral simply as a function of curves, we may regard $\int_{s_1} \log \frac{1}{r}\, d\, y_1$ in this way as defined for any curve which is the frontier of a simply

* The rest of this section may be omitted if we restrict ourselves to this subclass of curves of Γ.

† See [6]), p. 57.

connected region σ_1 consisting merely of interior points, or even for the complete frontier of any open set.[14]) But we may legitimately ask for the meaning of the integral with respect to a *single* curve: in what sense is this certain improper integral defined?

Consider a curve s_1 of class Γ. The integral is not an improper integral unless M is a point of s_1. In this case, let us write

$$q_n(M_1, M) = \log\frac{1}{r}, \ \log\frac{1}{r} \leq n,$$
$$= n, \text{ otherwise,}$$

and show that the limit of $\int_{s_1} q_n(M_1, M)\, d\, y_1$ is the value already obtained for the integral regarded as a function of curves or point sets. In fact, with M as center we construct a circle of radius

$$R_n = e^{-n}$$

and, regarding σ_1 as the open region of which s_1 is the boundary, denote by $\sigma_1^{(n)}$ the region obtained from σ_1 by subtracting all the points of the circle R_n including those of its boundary. Further, we denote by $s_1^{(n)}$ the frontier of the open set $\sigma_1^{(n)}$, of which M is an exterior point. Now since $q_n(M_1, M)$ is constant for points M_1 inside and on the boundary of R_n, we have:

$$\int_{s_1} q_n(M_1, M)\, d\, y_1 = \int_{s_1^{(n)}} q_n(M_1, M)\, d\, y_1 = \int_{s_1^{(n)}} \log\frac{1}{r}\, d\, y_1.$$

The region $\sigma_1^{(n)}$ consists of a finite number or a denumerable infinity of non-overlapping simply-connected regions, to all of which M is exterior, and therefore:

$$\int_{s_1^{(n)}} \log\frac{1}{r}\, d\, y_1 = \int_{\sigma_1^{(n)}} \frac{\cos x\, r}{r}\, d\, \sigma_1$$

[14]) P. J. Daniell, "Integrals around general boundaries," Bulletin of the American Mathematical Society, Vol. 25 (1918), pp. 65–68.

Hence, as n becomes infinite, we have

$$\lim_{n=\infty}\int_{s_1} q_n(M_1, M)\, d\, y_1 = \int_{\sigma_1} \frac{\cos x\, r}{r}\, d\, \sigma_1,$$

which was to be proved.

3.31. In order to treat the double integral in (12) for general regions it is first desirable to demonstrate a property of its integrand, viz., *the integral*

$$\int_{\sigma_1} \frac{\cos x\, r}{r}\, d\, \sigma_1 = \int_{s_1} \log \frac{1}{r}\, d\, y_1 = I(M)$$

represents a continuous function of M.

In fact if we form the difference for two neighboring points M, M', the point M′ being taken within a small circle of center M and radius R, and if we denote by σ^1 the part of σ_1 outside R, we have

$$| I(M') - I(M) | \leq \int_R \left| \frac{1}{r'} + \frac{1}{r} \right| d\, \sigma_1 + \left| \int_{\sigma'} \frac{\cos x\, r'}{r'}\, d\, \sigma_1 \right.$$

$$\left. - \int_{\sigma_1} \frac{\cos x\, r}{r}\, d\, \sigma_1 \right|.$$

The first term of the right-hand member of this inequality is obviously $\leq 6\pi R$ and therefore $< \frac{\epsilon}{2}$ if $R < \epsilon/12\pi$. The point M' is however within the circle; by taking it near enough to M the two integrals of the second term can be made to differ by as little as we please, say $\epsilon/2$, and therefore $| I(M') - I(M) |$ may be made $< \epsilon$. The continuity of $I(M)$ is thus established.

3.32. The identity (12) may now be treated. The right-hand member is merely the quantity

$$\text{(12}'') \qquad \int_\Sigma d\, f(e) \int_{\sigma_1} \frac{\cos x\, r}{r}\, d\, \sigma_1$$

of which the integrand is a continuous function of M, and which was discussed in connection with (10). If we consider this as a function of the open point set σ_1 it is again absolutely continuous, and thus the curvilinear integral of the left-hand member of (12), by means of the same extension as before, becomes a single valued function of curves. In fact, the quantity

$$\left|\int_{\sigma_1} \frac{\cos x\, r}{r}\, d\,\sigma_1\right| = \eta(\sigma_1)$$

may be made uniformly small with σ_1, even if σ_1 is allowed to consist of a denumerably infinite aggregate of simply connected open regions. We have

$$\eta(\sigma_1) \leq \int_{s_1} r\, d\,\theta_1,$$

where $d\,\sigma_1$ is expressed in terms of polar coördinates with center at M. But this integral can be split into two parts, one for which $r \leq r_0 \leq \epsilon/4\,\pi$, the other being extended over the remainder s' of s_1. Hence

$$\eta(\sigma_1) \leq \frac{\epsilon}{2} + \frac{2}{r_0}\int_{s'} \frac{r^2}{2}\, d\,\theta_1 \leq \frac{\epsilon}{2} + \frac{2\,\sigma}{r_0}$$

and this whole can be made $\leq \epsilon$, independently of the position of M, by taking $\sigma \leq \epsilon^2/16\pi^2$. Hence it follows that the expression (12″) in absolute value can be made arbitrarily small ($< \epsilon\; t(\Sigma)$ by making the measure of σ_1 small enough, and thus (12″) defines an absolutely continuous function of point sets.

The value of the left-hand member of (12) just obtained as a function of curves may also be obtained taking again a sort of principal value of a curvilinear integral not strictly convergent. In fact, with the previous definition of $q_n(M_1, M)$, we have

$$\lim_{n=\infty}\int_{s_1} d\,y_1 \int_{\Sigma} q_n(M_1, M)\; d\,p(e) = \int_{s_1} d\,y_1 \int_{\Sigma} \log\frac{1}{r}\, d\,p(e)$$

$$\lim_{n=\infty}\int_{s_1} d\,y_1 \int_{\Sigma} q_n(M_1, M)\; d\,n(e) = \int_{s_1} d\,y_1 \int_{\Sigma} \log\frac{1}{r}\, d\,n(e)$$

$$\lim_{n=\infty}\int_{s_1} d\,y_1 \int_{\Sigma} q_n(M_1, M)\; d\,f(e) = \int_{s_1} d\,y_1 \int_{\Sigma} \log\frac{1}{r}\, d\,f(e).$$

Equation (12) also remains true if for Σ we substitute an arbitrary σ bounded by a curve of class Γ and for $f(e)$ an $F(s)$, additive and of limited variation.

4. Functions of curves and functions of point sets. With the directions already assigned to n and r, the following evaluation is evident:

$$(13)\qquad \frac{1}{2\pi}\int_{s_1}\frac{\cos n\, r}{r}\,d\,s_1 = 0, \qquad M \text{ without } s_1,$$
$$= 1, \qquad M \text{ within } s_1,$$
$$= \theta,\ 0 \leq \theta \leq 1,\ M \text{ on } s_1.$$

If M is not a vertex on s_1, the quantity θ has the value ½; if M is a vertex, the value of θ measures the angle between the forward and backward tangents at M.

Given the additive function of point sets $f(e)$, let us define the function of curves $F(s_1)$

$$(14)\qquad F(s_1) = \int_{s_1} d\,s_1 \int_{\Sigma}\frac{\cos n\, r}{r}\,d\,f(e),$$

It is additive, and also of limited variation. In fact if $P(s_1)$ denotes the functional given by (14) when $f(e)$ is replaced by $p(e)$, and $N(s_1)$ corresponds similarly to $n(e)$, then by (11) and (13), $P(s_1)$ and $N(s_1)$ are not negative for any s_1, and by (14), $F(s_1)$ is seen to be additive and equal to $P(s_1) - N(s_1)$. Moreover, if by σ_1, we denote the set of points interior to s_1, by s_1' the set of points constituting s_1, except for the vertices, and by $M_1^{(i)}$ the vertices, equations (13) and (11) yield the formula

$$(14')\qquad F(s_1) = f(\sigma_1) + \frac{1}{2}f(s_1') + \Sigma_i\, \theta_i f(M_1^{(i)}).$$

Whenever a function of curves is given on curves of class Γ by a formula like (14) or (14′) we shall say that it has *discontinuities of the first kind.*

If a region σ_0 which contains a point M_0 as an interior point and has as boundary a curve s_0 of Γ is allowed to shrink in such a way that its diameter approaches zero, the quantity $F(s_0)$ approaches a definite limit $F(\overline{M}_0)$:

$$F(\overline{M}_0) = \lim_{\sigma=0} F(s_0),$$

since it is the difference of two quantities which are not negative. There can be only a denumerable infinity of points M^i such that $F(\overline{M}^i) \neq 0$.

If σ_n is a region such that its s_n consists only of interior points of σ_1, the quantity $F(s)$ approaches a definite limit

$$\lim F(\underline{s}_n) = F(\underline{s}_1),$$

as σ_n approaches σ_1, for the reason given just previously. In a similar manner by taking regions which contain all the points of s_1 as interior points, the quantity $F(\bar{s}_1)$ is defined.

In the case of the function of curves given by (14), we have

$$(14'') \qquad \begin{aligned} F(\underline{s}_1) &= f(\sigma_1) \\ F(\bar{s}_1) &= f(\sigma_1) + f(s_1) \\ F(\overline{M}_1^{(i)}) &= f(M_1^i) \end{aligned}$$

and therefore from (14′), the equation

$$(14''') \quad \begin{aligned} F(s_1) &= F(\underline{s}_1) + \frac{1}{2}\{F(\bar{s}_1) - F(\underline{s}_1)\} + \Sigma\left(\theta_i - \frac{1}{2}\right) F(\overline{M}_1^{(i)}) \\ &= \frac{1}{2}\{F(\underline{s}_1) + F(\bar{s}_1)\} + \Sigma\left(\theta_i - \frac{1}{2}\right) F(\overline{M}_1^{(i)}). \end{aligned}$$

Equations (14′) (14′′′) show that the notion of discontinuities of the first kind of a function of curves is the direct generalization of the corresponding concept for functions of a single variable.

4.1. We say that a function of curves $F(s)$ is *continuous at a curve* s when the condition

$$(15) \qquad F(\underline{s}_1) = F(\bar{s}_1)$$

is satisfied. We wish to discuss the correspondence between functions of curves $F(s_1)$, additive and of limited variation, and additive functions of point sets $f(e)$, subject to the equation

$$F(s) = f(\sigma_1) \tag{16}$$

which shall hold for any curve s_1 at which $F(s_1)$is continuous (*i.e.*, $f(s_1) = 0$).

Equations (16) imply equations (14″), and therefore, *given an additive function of point sets $f(e)$ there corresponds to it a unique function of curves, additive and of limited variation, provided that the discontinuities of the latter are merely of the first kind.* This function of curves is given by (14).

4. 2. In order to illustrate the correspondence (16) more fully, the following example may be considered. Let Σ be the square $0 < x < 1$, $0 < y < 1$; and to the points M^i of rational coördinates in this Σ assign positive numbers $p(M^i)$ in such a way that when the points are arranged in countable order the series of numbers will be convergent; then make the definition

$$f(e) = p(e) = \Sigma_i\, p(M^i), \tag{17}$$

where the summation is extended over all the M^i in the set e. Introduce now an auxiliary function $\mu(\theta, M)$ which for a given point M depends on an angle θ made with a fixed direction, and which for every M satisfies the condition $\int_0^{2\pi} \mu(\theta, M)\, d\,\theta = 1$. Consider the quantity

$$H(M, \theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \mu(\theta, M)\, d\,\theta.$$

For the purpose of defining $F(s)$, every point of s may be considered temporarily as a vertex, the tangents forward and backward from M making respectively angles θ_1, θ_2, with the fixed direction. Then the equation

$$F(s) = \Sigma_\sigma{}'\, p(M^i) + \Sigma_s\, H(M^i, \theta_1, \theta_2)\, p(M^i)$$

where the first summation takes all the points M^i interior to

σ and the second all the points M^i on the boundary s, satisfies the equation of correspondence (16), and defines an additive function of curves of limited variation which is discontinuous at every curve which contains a rational point. In general, these discontinuities will not be of the first kind; in order to make them so everywhere, and thus satisfy (14), it is sufficient to put $\mu(\theta, M) = \frac{1}{2\pi}$.

In general then the $F(s)$ is not uniquely determined by (16) and the $f(e)$. But in the converse case there is a unique determination.

4.3. *Given $F(s)$ additive and of limited variation, defined for the curves of Γ there is one and only one additive function of point sets $f(e)$ which satisfies the equation* (16). *It is defined for all point sets e measurable in the Borel sense.*

As we shall see in the course of the proof, it is sufficient if the class merely contains all the rectangles in Σ. Let w be an arbitrary rectangle whose sides are parallel to two fixed directions x, y. Let w' be the rectangle obtained by a displacement of w of amount ϵ in the direction of negative x and of amount η in the direction of negative y, ϵ and η being both positive quantities and so small that w' as well as w lies entirely within Σ. Then the limit

$$F'(w) = \lim_{\substack{\epsilon=0\\ \eta=0}} F(w')$$

exists, for $F(s)$ is the difference of two not negative functionals $P(s)$ and $N(s)$, and the region w' is formed as the difference of two regions each of which decreases with ϵ and η. Moreover, since the quantities $P(s)$, $N(s)$ can obviously be discontinuous merely on a denumerable infinity of curves, among a family of curves such that of any two curves one is entirely within the other, it follows that there are only a denumerable infinity of values of ϵ, η for which $F(w')$ is discontinuous.

Let ω' be the set of points interior to w' and ω the set of points

$$\omega = \lim_{\substack{\epsilon=0\\ \eta=0}} \omega'$$

and let M be the vertex of w of which the values of x, y are least. It is the only vertex of w which forms a point of ω.

If there is an $f(e)$ which satisfies the condition (16) it will satisfy the equation

$$\lim_{\epsilon,\,\eta=0} f(\omega') = f(\omega)$$

and therefore, as we see by choosing a sequence of pairs of values ϵ, η for which $F(w')$ is continuous,

$$f(\omega) = F'(\omega).$$

Hence the difference of two such functions will be zero for every ω.

In order to see that there is one such $f(e)$, we set up a $p(e)$ by means of $P(s)$ and an $n(e)$ by means of $N(s)$ and write $f(e) = p(e) - n(e)$. To determine such a $p(e)$ we establish the *weight* of every set ω with respect to it.

With the point M fixed, we let the diameter of ω approach zero, and define

$$p(M) = \lim_{\omega=0} P(\omega') = P(\overline{M})$$

a quantity which we have seen is zero except possibly for a denumerable infinity of points M^i. For an arbitrary ω, the *weight* of $p(e)$ is defined by the equation

$$p'(\omega) = P'(w) - \Sigma_i\, p(M^i),$$

where the summation is extended over all the points M^i contained in ω. We have $p'(M) = 0$ for every point M.

We know that there is one and only one function of point sets $p'(e)$, additive and not negative, which coincides with

$p'(\omega)$ when e reduces to ω.* It is continuous, as continuity is defined for functions of point sets ($p'(M)=0$ for every M), but not necessarily absolutely continuous, and determinate for every point set measurable in the Borel sense.

With the $p'(e)$ thus defined, we may now restore the discontinuities, and write

(17)
$$p(e) = p'(e) + \Sigma_i\, p(M^i),$$
$$f(e) = p(e) - n(e),$$

after $n(e)$ has been similarly determined. The equation (16) is seen to be satisfied for every curve s such that $f(s)=0$.

The determination of $f(e)$ is moreover unique, since the difference of two such functions, being zero for every ω, is zero, by the reasoning of the previous paragraph for every point set measurable (B).†

4.4. In relation to (14) we might consider also the equation

$$G(s_1) = \int_{s_1} d\, s_1 \int_s \frac{\cos n\, r}{r}\, d\, F(s)$$

where s_1 is constituted of points *within* s. This equation regularizes the discontinuities of $F(s)$, reducing them to discontinuities of the first kind and satisfying the equation

$$G(s_1) = F(s_1),$$

if $F(s_1)$ is continuous at s_1. On the other hand, if s_1 is made to coincide with s, the following equation replaces (14″ ′)

$$G(s) = F(s) + \frac{1}{2}\{F(s) - F(\underline{s})\} + \Sigma\left(\theta_i - \frac{1}{2}\right) F(M_s^i).$$

Here M_s^i are the vertices on s, and $F(M_s^i)$ denotes for a given vertex the portion of the value of $F(\overline{M}_s^i)$ which is assigned to $F(s)$ in its definition as a function of curves.

* See [6]), pp. 100–104.

† We shall write measurable (B) and measurable (L) to indicate measurability in the Borel and the Lebesgue senses, respectively.

5. Generalized derivatives and potentials. If $\phi(M)$ is a vector point function whose components in two fixed directions, say x and α, are summable superficially, its component in any fixed direction β will be summable superficially. Likewise if $u(M)$ is a scalar point function which may be integrated with respect to x and with respect to α along any curve of class Γ, it may be integrated with respect to the coördinate parallel to an arbitrary direction β; *i.e.*, $\int_s u\, d\,\beta$ exists.

Let $\phi(M)$ be a vector point function whose component ϕ_α in every fixed direction α is summable superficially, and $u(M)$ be a scalar point function summable superficially and such that $\int u(M)\, d\,\alpha$ may be defined for every given direction α. Then $\phi(M)$ is spoken of as a *gradient* of $u(M)$ and $u(M)$ as a *potential function* of $\phi(M)$ provided the equation

$$\int_\sigma \phi_\alpha\, d\,\sigma = \int_s u\, d\,\alpha' \tag{18}$$

is satisfied for every s of Γ and for every fixed direction α, the direction α' being fixed and in advance of α by $\pi/2$.

It is sufficient that (18) should hold for two distinct directions α_1, α_2 in order to hold for every direction α.

It is legitimate to ask in what sense the component of $\phi(M)$ in the direction α is the derivative of $u(M)$ in the direction α. The directional derivative of a scalar point function is not necessarily a vector; in other words, the formula

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial x} \cos x\, \alpha + \frac{\partial u}{\partial y} \cos y\, \alpha$$

does not hold simply on the basis of the existence of $\dfrac{\partial u}{\partial x}$

and $\frac{\partial u}{\partial y}$. If however this derivative for two directions, say $\partial u/\partial x$ and $\partial u/\partial y$, is summable superficially and satisfies for these two directions and all curves of Γ the equation (18), then the vector which has these two quantities for its components will be a gradient vector, and the function $u(M)$ its potential function. Nevertheless it is simpler than this and less artificial to use directly a generalized partial derivative which has itself vectorial properties.

5.1. We say that $D_\alpha u$, *the generalized derivative in the direction* α *of* $u(M)$, is the limit, if such limit exists, of the expression

$$D_\alpha u = \lim_{\sigma=0} \frac{1}{\sigma} \int_s u \, d\alpha' \tag{19}$$

where the fixed direction α' makes an angle $\pi/2$ with the fixed direction α, and σ denotes the area enclosed by s; it is understood that σ tends towards 0 in such a way that the ratio σ/d^2, where d is the diameter of σ, remains different from 0 by some positive quantity.

If at a point M, the generalized derivative exists for two distinct directions α_1, α_2, it exists for any other direction α, since the contour integral with respect to $d\alpha'$ can be evaluated in terms of those with respect to $d\alpha_1'$ and $d\alpha_2'$, in fact $d\alpha'$ bears the same linear relation to $d\alpha_1'$ and $d\alpha_2'$ as $d\alpha$ to $d\alpha_1$ and $d\alpha_2$ and this relation is independent of the variable of integration in the contour integral. Hence in particular

$$D_\alpha u = D_x u \cos x\alpha + D_y u \cos y\alpha \tag{20}$$

where

$$D_x u = \lim_{\sigma=0} \frac{1}{\sigma} \int_s u \, dy, \quad D_y u = -\lim_{\sigma=0} \frac{1}{\sigma} \int_s u \, dx.$$

5.2 It is now possible to see the relation between generalized derivative and gradient, although this again necessitates a return to general ideas. If a function $\psi(M)$ is integrable superficially, its integral

$$\int_\sigma \psi(M)\, d\,\sigma$$

generates an *absolutely continuous* function of point sets $f(e)$. In connection with $f(e)$, the quantity

$$f'(M) = \lim_{\sigma=0} \frac{f(\sigma)}{\sigma}$$

where σ approaches zero in the manner just described, about the point M, defines what may be called a *total* or *superficial* derivative of $f(e)$. A similar definition is obviously applicable to an additive function of curves, and in fact merely specializes slightly the well-known definition of Volterra of derivative of a functional, by specifying the manner in which σ must approach zero. According to our definitions, if we write

$$F(s) = \int u\, d\, y$$

we shall have

$$D_x\, u(M) = F'(M)$$

and similarly for any direction α. Now it is a well-known theorem, due to Lebesgue, that an absolutely continuous function of point sets possesses a superficial derivative except possibly at the points of a set of measure zero, moreover that it is itself the superficial integral of that derivative; that is, in the case given,

$$f'(M) = \psi(M),$$

almost everywhere.* Hence if a vector $\phi(M)$ is a gradient of $u(M)$, according to our definition, it follows from the theorem of Lebesgue that *for any direction α the equation*

(21) $$D_\alpha\, u = \phi_\alpha,$$

* By *almost everywhere* we shall mean, except possibly for a point set of measure zero, superficial or linear, according to the context.

is valid except possibly at the points of a set of superficial measure zero; moreover it follows directly from equation (20) that *this set may be chosen so as to be independent of the direction* α.

5.3 We are in a position to establish connection with partial derivatives in the usual sense. This is done by means of theorems which enable us to integrate the generalized partial derivatives with respect to curvilinear coördinates, θ, χ and evaluate such expressions as

$$\int \frac{1}{J}\left\{ D_x u \frac{\partial x}{\partial \chi} + D_y u \frac{\partial y}{\partial \chi} \right\} d\sigma$$

where J denotes the quantity

$$J = \frac{d(\theta\, \chi)}{d(x\, y)}.$$

5.31. Consider a rectangle with vertices (x_0, y_0), (x, y) and a function $u(x, y)$ which is the potential of its generalized derivatives. The formula

$$\int_{y_0}^{y} \{u(x, \eta) - u(x_0, \eta)\}\, d\eta = \int_{y_0}^{y} d\eta \int_{x_0}^{x} D_x\, u(\xi, \eta)\, d\xi$$

is therefore valid. On a given line $x = \text{const.}$, the function $u(x, y)$ is the derivative of its own indefinite integral in the Lebesgue sense almost everywhere; that is, since $u(x, y)$ is measurable superficially, unless (x, y) lies in a certain set of superficial measure zero we may write:

$$u(x, y) = \frac{\partial}{\partial y} \int_{y_0}^{y} u(x, \eta)\, d\eta,$$

and $$u(x, y) - u(x_0, y) = \frac{\partial}{\partial y} \int_{y_0}^{y} \{u(x, \eta) - u(x_0, \eta)\}\, d\eta,$$

unless (x, y) or (x_0, y) lie in this set. But also, with reference to an important theorem of analysis,[15]), since $D_x u$ is summable superficially, the quantity $\int_{x_0}^{x} D_x\, u(\xi, y)\, d\xi$ is the

[15]) C. de la Vallée Poussin, "Cours d'Analyse Infinitésimale," Paris (1912), Vol. II, p. 122.

derivative of its integral with respect to y, given x and x_0, for almost all y, that is, unless (x, y) or (x_0, y) belong to a certain set of measure zero.

Hence there is a set E_1 of superficial measure zero, such that if neither (x, y) nor (x_0, y) lie in it, the equation

$$u(x, y) - u(x_0, y) = \int_{x_0}^{x} D_x\, u(\xi, y)\, d\,\xi$$

is valid. There is a similar set E_2 of measure zero, such that if (x_0, y) and (x_0, y_0) do not lie in it, we shall have

$$u(x_0, y) - u(x_0, y_0) = \int_{y_0}^{y} D_y\, u(x_0, \eta)\, d\,\eta.$$

We wish to show that we can so choose (x_0, y_0) that the equation

$$u(x, y) - u(x_0, y_0) = \int_{y_0}^{y} D_y\, u(x_0, \eta)\, d\,\eta + \int_{x_0}^{x} D_x\, u(\xi, y)\, d\,\xi$$

will hold almost everywhere in (x, y).

Let e be the set of points (x, y) composed of lines $x =$const., which contain points of E_1 or E_2 that form sets not of linear measure zero; since $u(x, y)$ is measurable superficially, the set e is measurable and is of superficial measure zero. Let (x_0, y_0) be any fixed point not in $E_2 + e$ (of superficial measure zero), and let the set of points (x, y) for which (x_0, y) lies in E_1 or E_2, be denoted by e'. This set e' is also of superficial measure zero, by reason of the choice of x_0 with respect to e. The desired equation is now seen to hold unless (x, y) lies in the set of superficial measure zero, $E = E_1 + e'$.

Hence, $u(x, y)$ being a potential function for its generalized derivatives, the function $\bar{u}(x, y)$

$$\bar{u}(x, y) = u(x_0, y_0) + \int_{y_0}^{y} D_y\, u(x_0, \eta)\, d\,\eta + \int_{x_0}^{x} D_x\, u(\xi, y)\, d\,\xi, \tag{22}$$

will differ from $u(x, y)$ only in the points of E; moreover, *almost everywhere in the rectangle* the derivative in the usual sense $\partial\bar{u}/\partial\, x$ exists and is identical with $D_x\, u$.

5.311. If $u(x, y)$, a potential function of its generalized derivatives, is itself continuous, the function $\bar{u}(x, y)$ will be identical with $u(x, y)$ except for possible values of y that constitute a null set linearly and $\partial u/\partial x$ and (in a similar manner) $\partial u/\partial y$ will exist and will equal $D_x u$ and $D_y u$ respectively *almost everywhere.*

5.32. The function $\bar{u}(x, y)$ it may be noted is for any value of y (except for those which may constitute a set of linear measure 0) an absolutely continuous function of x. Consider also a second function $v(x, y)$ continuous in (x, y) and absolutely continuous in x for every value of y. Except for the exceptional values of y we can write

$$\int_{x_1}^{x_2} v(\xi, y) \frac{\partial \bar{u}(\xi, y)}{\partial \xi} d\xi = \Big[v(\xi, y)\, \bar{u}(\xi, y) \Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} \bar{u}(\xi, y) \frac{\partial v(\xi, y)}{\partial \xi} d\xi,$$

and therefore for any points (x_1, y) (x_2, y) which do not contain these exceptional values of y nor lie in the point set E previously described — let us denote this combined set of points of superficial measure zero by E' — we have the equation

$$\int_{x_1}^{x_2} v(\xi, y)\, D_x u(\xi, y)\, d\xi = \Big[v(\xi, y)\, u(\xi, y) \Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} u(\xi, y) \frac{\partial v(\xi, y)}{\partial \xi} d\xi.$$

Hence for any curve s_1 of Γ which does not contain of E' points which constitute on s_1 a set of linear measure different from zero, we have the identity

$$\int_{\sigma_1} v\, D_x u\, d\sigma = \int_{s_1} v\, u\, dy - \int_{\sigma_1} u \frac{\partial v}{\partial x} d\sigma;$$

in fact u and therefore $v\, u$ may be integrated with respect to y round any curve of Γ.

In order to establish this identity in its true generality, that is, for any curve of Γ, it is necessary to investigate the curvilinear integral in more detail. For this purpose, consider a small neighborhood, which we might as well take to be the neighborhood of the origin, and define the function $p(x, y)$

$$p(x, y) = \int_0^x | D_x u | \, d x,$$

which is absolutely continuous in x except for values of y which form at most a point set of zero measure. For any curve s of Γ and its σ, we have then

$$\int_s p \, d y = \int_\sigma | D_x u | \, d \sigma,$$

so that $\int_s p \, d y$ represents a positive absolutely continuous function of curves.

Let now s' be a portion of a curve s and s_δ' be the same portion displaced by an amount $-\delta$ in the direction of the x-axis, and let s_δ be the closed curve formed of s', s_δ' and two portions parallel to the x-axis. We have now the result:

5.321. If $v(x, y)$ is a continuous function of x and y, then

$$\lim_{\delta=0} \int_{s_\delta} v(M) \, p(M) \, d y = 0, \tag{22'}$$

or, what is the same thing,

$$\lim_{\delta=0} \int_{s_\delta'} v(M) \, p(M) \, d y = \int_{s'} v(M) \, p(M) \, d y. \tag{22''}$$

In fact, we may write the difference between the two members in the form

$$\int_{M_1}^{M_2} \{v(x, y) \, p(x, y) - v(x_\delta, y) \, p(x_\delta, y)\} \, d y =$$

$$\int_{M_1}^{M_2} v(x_\delta, y) \, \{p(x, y) - p(x_\delta, y)\} \, d y +$$

$$\int_{M_1}^{M_2} \{v(x, y) - v(x_\delta, y)\} \, p(x, y) \, d y,$$

and each of these two integrals may be made as small as we please with δ, for not only is $p(x, y)$ itself positive or zero everywhere, but also the quantity

$$\int_{M_1}^{M_2} \{p(x, y) - p(x_\delta, y)\}\, dy$$

is positive or zero, by definition, and no matter where the point M_2 may be, on the curve s'. Hence we deduce

$$p(x, y) - p(x_\delta, y) \geqq 0$$

for *almost all* values of y. We can then apply the law of the mean to the two integrals, and write the numerical value of the sum as

$$\leq M \int_{s_\delta} p(x, y)\, dy + \omega_\delta \int_{s'} p(x, y)\, dy,$$

where M is the maximum value of $|\, v(x, y)\, |$, and ω_δ the maximum value of the oscillation of $v(x, y)$. The first term is $\leq M\, t_p(\sigma_\delta)$ and the integral of the second term is independent of δ, whence we can draw the desired conclusion.

Now the function

$$n(x, y) = -u(x, y) + p(x, y)$$

is a function of the same sort as $p(x, y)$ and the same theorem applies to it, and therefore the same theorem applies to $u(x, y)$ itself. That is to say, we may substitute $u(x, y)$ for $p(x, y)$ in the equations (22′) and (22″).

If we return to the identity of Art. 5.32, we can form the curve s_δ corresponding to an arbitrary s *of* Γ, and divide the curve s into portions $s^{(i)}$ small enough and take δ small enough so that each portion and the corresponding portion of s_δ lies within a square which itself lies entirely within Σ. We see therefore that the quantity

$$\int_{s_\delta} u\, v\, dy,$$

is a continuous function of δ, and since also the other terms of the identity of Art. 5.32 are absolutely continuous functions of point sets, it follows that since the identity is valid for *almost all* values of δ it is true for all of them. The following theorem is therefore demonstrated.

5.322. If $u(M)$ is a potential function for its generalized derivatives, and s is any curve of Γ, and if $v(M), w(M)$ are two functions continuous in Σ, the function v being absolutely continuous in x for every y and the w being absolutely continuous in y for every x, then the following identities are valid.

$$\int_\sigma v D_x u \, d\sigma = \int v u \, dy - \int_\sigma u \frac{\partial v}{\partial x} d\sigma \tag{23}$$

$$\int_\sigma w D_y u \, d\sigma = -\int w u \, dx - \int_\sigma u \frac{\partial w}{\partial y} d\sigma.$$

5.33 Theorem. *Let $x = x(\chi, \theta)$, $y = y(\chi, \theta)$ be continuous with their first partial derivatives in Σ and let $J = d(x, y)/d(\chi, \theta)$ be different from zero in this region; further, let $u(M)$ be a potential function for its generalized derivatives in Σ. For any curve s of Γ the equations*

$$\int_\sigma \frac{1}{J}\left\{D_x u \frac{\partial x}{\partial \chi} + D_y u \frac{\partial y}{\partial \chi}\right\} d\sigma = \int_{\bar{s}} u \, d\theta \tag{24}$$

$$\int_\sigma \frac{1}{J}\left\{D_x u \frac{\partial x}{\partial \theta} + D_y u \frac{\partial y}{\partial \theta}\right\} d\sigma = -\int_{\bar{s}} u \, d\chi$$

are valid, with $\bar{s}$ understood as the transform of s in the χ, θ plane; and for any point (χ, θ) which is the transform of a point M which does not belong to a certain fixed set of zero measure, the derivatives $D_\chi u$ and $D_\theta u$ exist in the transformed plane, and the formulae

$$D_\chi u = D_x u \frac{\partial x}{\partial \chi} + D_y u \frac{\partial y}{\partial \chi} \tag{25}$$

$$D_\theta u = D_x u \frac{\partial x}{\partial \theta} + D_y u \frac{\partial y}{\partial \theta}$$

are valid.

The equations (25) provide an extension of the vector law for the generalized derivatives, in terms of curvilinear coördinates. When it is necessary to distinguish between $D_\alpha u$ and these generalized derivatives with respect to curvilinear coördinates we shall indicate the latter by the symbols $(D_\chi u)_{\chi\theta}$ etc. If the curvilinear coördinates are orthogonal, and $\theta=$const. at a point M gives the direction α it is evident that at the point M

$$(D_\chi u)_{\chi\theta}=D_\alpha u\,\frac{d\alpha}{d\chi},$$

and there is no ambiguity in the symbol $D_\chi u$, even when $d\alpha=d\chi$.

In order to prove the theorem, assume first that the quantities

$$\frac{\partial\theta}{\partial x},\ \frac{\partial\theta}{\partial y},\ \frac{\partial\chi}{\partial x},\ \frac{\partial\chi}{\partial y},$$

are absolutely continuous in x for every y and in y for every x. Then, since

$$J\frac{\partial\chi}{\partial x}=\frac{\partial y}{\partial\theta},\quad J\frac{\partial\chi}{\partial y}=-\frac{\partial x}{\partial\theta},$$
$$J\frac{\partial\theta}{\partial x}=-\frac{\partial y}{\partial\chi},\quad J\frac{\partial\theta}{\partial y}=\frac{\partial x}{\partial\chi},$$

and since therefore

$$\int\frac{1}{J}\left\{D_x u\,\frac{\partial x}{\partial\chi}+D_y u\,\frac{\partial y}{\partial\chi}\right\}d\sigma=\int\left\{D_x u\,\frac{\partial\theta}{\partial y}-D_y u\,\frac{\partial\theta}{\partial x}\right\}d\sigma,$$

it follows by the theorem of Art. 5.322, that

$$\int_\sigma\frac{1}{J}\left\{D_x u\,\frac{\partial x}{\partial\chi}+D_y u\,\frac{\partial y}{\partial\chi}\right\}d\sigma=\int_s u\left(\frac{\partial\theta}{\partial y}dy+\frac{\partial\theta}{\partial x}dx\right)$$
$$=\int_{\bar{s}} u\,d\theta.$$

As far as the final formulae are concerned, the conditions of absolute continuity on the partial derivatives are unnecessary. It is sufficient if these derivatives are merely continuous functions of x, y. The validity of this statement is assured by the proof given by C. de la Vallée Poussin, by a method of polynomial approximations, of a formula entirely analogous to this.*

In a similar manner, the second of the equations (24) is demonstrated.

With respect to the equations (25) we except the points of a set E_0 for which one of the quantities $D_x u$, $|D_x u|$, $D_y u$, $|D_y u|$ fails to be the superficial derivative of its own integral. If v is a continuous function of (x, y), the quantities $v D_x u$, $|v D_x u|$ etc. are the superficial derivatives of their own integrals at all except the points of E_0. In fact, if M_1 is not in E_0,

$$\left|\frac{1}{\sigma}\int v D_x u\, d\sigma - \frac{v(M_1)}{\sigma}\int D_x u\, d\sigma\right|$$
$$\leq \frac{1}{\sigma}\int |v(M) - v(M_1)|\,|D_x u|\, d\sigma$$
$$\leq \frac{\omega}{\sigma}\int |D_x u|\, d\sigma,$$

where ω is the maximum oscillation of $v(M)$ in σ.
Hence

$$\lim_{\sigma=0} \frac{\int v D_x u\, d\sigma}{\sigma} = v(M_1) \lim_{\sigma=0} \frac{1}{\sigma}\int D_x u\, d\sigma,$$

and similarly for the other expressions.

Except for points in E_0, we have therefore

$$\lim_{\bar{\sigma}=0} \frac{1}{\bar{\sigma}}\int_{\bar{s}} u\, d\theta = \lim_{\sigma=0} \frac{J}{\sigma}\int_{\bar{s}} u\, d\theta = D_x u \frac{\partial x}{\partial \chi} + D_y u \frac{\partial y}{\partial \chi},$$

by means of (24). Similarly the second of equations (25) is obtained.

The set E_0 is seen to be not only independent of directions α, but also even of the choice of curvilinear coördinates.

* See [15]), vol. 2, p. 24.

5.331. A further consequence of equations (24), (25) is that on *almost any* curve $\chi = \chi_0$, the equation

$$\int_{\theta_1}^{\theta_2} D_\theta u(\chi_0, \theta)\, d\theta = u(\chi_0, \theta_2) - u(\chi_0, \theta_1) \tag{25'}$$

is valid. A similar equation holds for $D_\chi u$ on curves $\theta = \text{const.}$

6. Integro-differential equations of Bôcher type. Laplace's equation. Lest our analysis seem sterile, we shall interrupt its development at this point in order to consider some of its applications. We shall discuss the generalization of Poisson's equation and show its relation to Laplace's equation in the usual form.

Consider the equation

$$\int_{s_1} \phi_n\, d s_1 = F(s_1), \tag{26}$$

in which $F(s_1)$ is a function of curves, additive and of limited variation, whose discontinuities are of the first kind.

A vector ϕ which is a solution of this equation is given by the equation

$$\phi_\alpha = \frac{1}{2\pi} \int_\Sigma \frac{\cos \alpha r}{r}\, d f(e). \tag{27}$$

By means of equations (11), (14), it is seen that *this vector is a solution of* (26) *for every curve* s_1 *of* Γ *in* Σ.

6.1. Since the right-hand member of (12) is (12″), also the right-hand member of (10), equations (10) and (12) yield the result:

$$\int_{\sigma_1} d\sigma_1 \int_\Sigma \frac{\cos \alpha r}{r}\, d f(e) = \int_{s_1} d\alpha' \int_\Sigma \log \frac{1}{r}\, d f(e),$$

the direction of α' being $\pi/2$ in advance of that of α. Hence the function

$$u(M_1) = \frac{1}{2\pi} \int_\Sigma \log \frac{1}{r}\, d f(e) \tag{28}$$

is a potential function for the vector ϕ of (27), and we have

$$\phi(M_1) = \nabla u(M_1). \tag{28'}$$

Hence *the function given by* (28) *is a solution of* (2) *for every curve* s_1 *of* Γ *provided* $F(s_1)$, *which is additive and of limited variation, has discontinuities merely of the first kind.*

6.2. Moreover, the equation

$$D_\alpha u = \phi_\alpha(M)$$

is valid except possibly at the points of a set E_0 (the E_0 of Art. 5.33) of superficial measure zero, which is the same for every α; and therefore, *for any curve* s_1 of Γ *which does not contain points of* E_0 *of more than linear measure zero, the function* $u(M)$ *is a solution of the equation*

$$\int_{s_1} D_n u(M_1)\, d s_1 = F(s_1). \tag{29}$$

Of course the curves on which $F(s)$ may be discontinuous may in the totality of their points constitute a set of more than zero measure; thus in the example given in Art. 4.2 any curve through a rational point in Σ is a curve of discontinuity.

The difference of two solutions of (2) is a solution of (1). It is important therefore to establish the relation of solutions of the latter equation to Laplace's equation $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{30}$$

In this case we have an extension of a well-known theorem of Bôcher.*

6.3 Theorem. *If* $u(M)$ *is a potential function for its gradient vector* ∇u, *and if the equation*

$$\int \nabla_n u\, d s = 0, \tag{1}$$

is satisfied for every s *of* Γ *in* Σ, *then the function* u (M) *has merely unnecessary discontinuities, and when these are removed by changing the value of* $u(M)$ *at most in the points of a set of superficial measure zero, the resulting function has*

* See [2]).

continuous derivatives of all orders and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \tag{30}$$

at every point.

Let us refer the point M to polar coördinates r, θ taking a fixed point M_1 as the pole and an arbitrary direction through M as an axis of reference. By means of the formulae (20), (25) we have

$$(D_r u)_{r\theta}=D_r u=D_x u\frac{x-x_1}{r}+D_y u\frac{y-y_1}{r},$$

and from (24)

$$\int_\sigma \frac{1}{r} D_r u\, d\sigma=\int_s u\, d\theta,$$

for any curve s of Γ which does not include M_1 in its interior. We take as the region σ the portion of the plane between two circles of radii $r=R_1$ and $r=R$ and center M_1, and since

$$\int_\sigma \frac{1}{r} D_r u\, d\sigma=\int_{R_1}^{R} d\,r\int_0^{2\pi} D_r u\, d\theta,$$

we see by (1) that the integral vanishes; for on the family of curves r const. we have

$$\nabla_n u=D_r u,$$

except at the points of a set of superficial measure zero. Hence

$$\int_0^{2\pi} d\theta\int_{R_1}^{R} D_r u\, d u=0,$$

and

$$\int_{r=R_1} u\, d\theta=\int_{r=R} u\, d\theta. \tag{31}$$

We can at this point let R_1 approach zero. The function $u(M)$, since it is summable superficially, is the superficial derivative of its superficial integral everywhere except possibly at the points of a set of measure zero. Let us denote this derivative by $\bar{u}(M)$. We have, for the circle of radius R_1

$$\frac{1}{\sigma}\int_\sigma u\,d\sigma=\frac{1}{\pi R_1^2}\int_0^{R_1} r\,dr\int_0^{2\pi} u\,d\theta$$
$$=\frac{1}{2\pi}\int_{r=R} u\,d\theta,$$

and therefore if $\bar{u}(M_1)$ exists

$$\bar{u}(M_1)=\frac{1}{2\pi}\int_{r=R} u\,d\theta. \tag{31'}$$

6.31. Let now M_1 be a point of the circle not its center, and refer to it the position of a point M by polar curvilinear coördinates (χ,ψ), where the curves $\psi=const.$ are the circles through M orthogonal to the circle R and the curves $\chi=const.$ are the orthogonal trajectories of the former circles; in order to make matters definite, let the χ of M be the value of the Green's function at m whose pole is at M_1 and which vanishes on the circle R and let the ψ of M be the value there of the conjugate function to the Green's function. Take as a region σ' that between the two circles $\chi=\chi_1$ and $\chi=0$ and form the integral

$$\int_{\sigma'}\frac{1}{J}D_\chi u\,d\sigma'=\iint D_\chi u\,d\chi\,d\psi=\int_{s'} u\,d\psi.$$

This integral may also be written in the form

$$\int_0^{\chi_1} d\chi\int_0^{2\pi} D_\chi u\,d\psi.$$

Here, however, since the coördinates are orthogonal

$$D_\chi u=D_n u\,\frac{d s_\chi}{d\chi}$$
$$d\psi=d s_\psi\,\frac{1}{\dfrac{d s_\psi}{d\psi}},$$

and since the χ, ψ represent the Green's function and its conjugate, the infinitesimal coördinate rectangles become squares, and

$$\left|\frac{d\,s_\chi}{d\,\chi}\right| = \left|\frac{d\,s_\psi}{d\,\psi}\right|,$$

whence the integral is zero. In fact:

$$\int D_n\,u\,d\,s_\psi = 0$$

for *almost all* values of χ.

Hence

$$\int_{x=x_1} u\,d\psi = \int_{x=0} u\,d\psi, \tag{32}$$

and if $\bar{u}(M_1)$ exists,

$$\bar{u}(M_1) = \frac{1}{2\pi}\int_{x=0} u\,d\psi. \tag{32'}$$

That is to say, $\bar{u}(M)$ is given where it exists by the Poisson's integral extended over the circle R, and this same value applies to the function $u(M)$ *almost everywhere* within R.

But the function given within R by Poisson's integral is itself continuous, and therefore the function $\bar{u}(M)$ does exist at all points within R, and consequently at any point M_1 in Σ. Moreover the continuous function given within R by Poisson's integral has continuous derivatives of all orders and satisfies (30); this fact then applies to $\bar{u}(M_1)$, at any point M_1 of Σ. But we can replace $u(M)$ by $\bar{u}(M)$ merely by changing the value of $u(M)$ at most in the points of a set of superficial measure zero, and the theorem is demonstrated.

7. Pseudo-polynomial approximations. Consider points (x, y) in a simple closed region Σ' entirely interior to Σ. Let Σ'' be another simple closed region similarly contained in Σ and containing all the points of Σ' (which includes its boundary) as interior points. Finally choose ϵ so small that any point of Σ', displaced in any direction by an amount less or equal to $\epsilon\sqrt{2}$, remains within Σ''.

Corresponding to any function $\phi(x, y)$ summable over Σ we define the function $\phi^{(m)}(x, y)$

(33) $\phi^m(x, y) =$

$$\frac{1}{k_m^2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\phi(x+\xi, y+\eta)\,(1-\xi^2)^m\,(1-\eta^2)^m\,d\xi\,d\eta,$$

with

$$k_m = \int_{-1}^{+1}(1-t^2)^m\,dt.$$

This function is not a polynomial, but is for our purposes a useful approximation for $\phi(x, y)$. In fact, under the supposed hypothesis that $\phi(x, y)$ is summable superficially in Σ it is well known that $\phi^{(m)}(x, y)$ converges to $\phi(x, y)$ as n becomes infinite, everywhere except possibly at the points of a set of superficial measure zero,* and uniformly over any region, which is closed and at every point of which $\phi(\psi, y)$ is continuous.

The function $\phi^{(m)}(x, y)$ is moreover a continuous function of x, y. The proof of this statement for two dimensions follows exactly that for one dimension,† and it is therefore unnecessary to reproduce it at this point.

Consider now a function $u(x, y)$ continuous in Σ and a potential function for its generalized derivatives. By Art. 5.311, *almost everywhere* $\partial u/\partial x$ and $\partial u/\partial y$ exist; and each is almost everywhere equal to the corresponding generalized derivative. If then we write $u_1(x, y) = \partial u(x, y)/\partial x$ where the latter exists and zero otherwise, and put

$$u_1^{(m)}(x, y) = \frac{1}{k_m^2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}u_1(x+\xi, y+\eta)\,(1-\xi^2)^m\,(1-\eta^2)^m\,d\xi\,d\eta,$$

* Tonelli's theorem. See [15]), vol. II, p. 135

† See [15]), vol. II, p. 163.

with similar expressions corresponding to $u_2(x, y) = \partial u(x, y)/\partial y$ and to $u(x, y)$ itself, we have almost everywhere

$$\lim_{m=\infty} u_1^{(m)}(x, y) = u_1(x, y),$$
$$\lim_{m=\infty} u_2^{(m)}(x, y) = u_2(x, y),$$

and everywhere

$$\lim_{m=\infty} u^{(m)}(x, y) = u(x, y), \text{ uniformly in } \Sigma'.$$

But everywhere in Σ' we have

(34)
$$\frac{\partial u^{(m)}(x, y)}{\partial x} = u_1^{(m)}(x, y),$$
$$\frac{\partial u^{(m)}(x, y)}{\partial y} = u_2^{(m)}(x, y),$$

the right-hand members being continuous functions of x, y.

Indeed, since for every ξ, η we have

$$\int_{\Sigma'} | u_1(x+\xi, y+\eta) | \, d\sigma < \int_{\Sigma''} | u_1(x, y) | \, d\sigma$$

which is some constant C independent of ξ, η it may be deduced that

$$\int_\sigma u_1^{(m)}(x, y) \, d\sigma = \frac{1}{k_m^2} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} d\xi \, d\eta (1+\xi^2)^m (1-\eta^2)^m \int_\sigma u_1(x+\xi, y+\eta) \, d\sigma,$$

and hence

$$\int_\sigma u_1^{(m)}(x, y) \, d\sigma = \int_s u^{(m)} \, dy,$$

so that $u^{(m)}$ is the potential function of its derivatives $u_1^{(m)}(x, y)$, $u_2^{(m)}(x, y)$. Moreover we have the equations

$$\int_\sigma u_1(x, y) \, d\sigma = \int_s u(x, y) \, dy = \lim_{m=\infty} \int_s u^{(m)}(x, y) \, dy$$
$$= \lim_{m=\infty} \int_\sigma u_1^{(m)}(x, y) \, d\sigma$$

$$\int_\sigma u_2(x, y) \, d\sigma = -\int_s u(x, y) \, dx = \lim_{m=\infty} - \int_s u^{(m)}(x, y) \, dx$$
$$= \lim_{m=\infty} \int_\sigma u_2^{(m)}(x, y) \, d\sigma.$$

We wish however to be able to investigate such limits as the following:

$$\lim_{m=\infty}\int_\sigma \phi(x, y)\, u_1^{(m)}(x, y)\, d\sigma \text{ and } \lim_{m=\infty}\int_s \phi(x, y)\, u^{(m)}(x, y)\, ds.$$

The following theorem may be demonstrated.

7.1. *If $u(x, y)$ is continuous in Σ and a potential function for its generalized derivatives, and if $\phi(x, y)$ is summable superficially, then*

(a) *For any curve of Γ in Σ'*

$$\lim_{m=\infty}\int_s \phi(x, y)\, u^{(m)}(x, y)\, ds = \int_s \phi(x, y)\, u(x, y)\, ds, \tag{35}$$

provided that $\phi(x, y)$ is summable along that curve, and

(b) *For any region σ bounded by a curve of Γ in Σ'.*

$$\lim_{m=\infty}\int_\sigma \phi(x, y)\, u_1^{(m)}(x, y)\, d\sigma = \int_\sigma \phi(x, y)\, u_1(x, y)\, d\sigma, \tag{36}$$

provided that either $\phi(x, y)$ or $u_1(x, y)$ remains finite or that both $\phi(x, y)$ and $u_1(x, y)$ are summable superficially with their squares.

The first part of the theorem is demonstrated immediately since the convergence of $u^{(m)}$ to u is uniform over Σ'.

As for the second part, under the hypotheses given, the quantity

$$\int_\sigma |\, u_1(\xi+x, \eta+y)\, \phi(x, y)\, |\, d\sigma$$

exists, and remains finite for values of ξ, η in the neighborhood of (0, 0); and moreover the quantity

$$\psi(\xi, \eta) = \int_\sigma u_1(\xi+x, \eta+y)\, \phi(x, y)\, d\sigma$$

represents a continuous function of ξ, η in the neighborhood of (0, 0).* Hence if we denote by $I_1^{(m)}$ the integral of the left-hand member of (36), we have

$$I_1^{(m)} = \frac{1}{k_m^2}\iint \psi(\xi, \eta)\, (1-\xi^2)^m\, (1-\eta^2)^m\, d\xi\, d\eta.$$

* As in [15]), vol. II. p. 163.

But, precisely as in the case of polynomial approximation, by writing

$$\int_{-\epsilon}^{\epsilon} = \int_{-\epsilon}^{-\epsilon'} + \int_{-\epsilon'}^{\epsilon'} + \int_{\epsilon'}^{\epsilon}$$

with $\epsilon' < \epsilon$, it is seen that $I_1^{(m)}$ tends towards the value of $\psi(0, 0)$ as a limit, and therefore (36) is proved.

2. *Green's Theorem*

8. Introduction. In this chapter we consider a scalar function $v(M)$ and a vector function $\phi(M)$ which is a solution of the equation

$$\int \phi_n \, d\, s = F(s), \tag{37}$$

n being the interior normal, for all curves of class Γ; the functional $F(s)$ is an additive function of curves of limited variation. Throughout this section $\phi(M)$ shall satisfy Condition N.

Condition N. Any component of ϕ is summable superficially, and the normal component of ϕ is summable along any curve of Γ; in particular, given s_0 there is an N such that if the length s of the curve is $< s_0$ the quantity $\int_s |\phi_n(M)| \, d\, s$ is $<N$; this whether s consists of a single curve of class Γ or a finite number of such curves, mutually exterior.

A special case of Condition N is the following:

Any component of ϕ is summable superficially, and the normal component of ϕ is summable along any curve of Γ; further, ϕ is supposed to remain finite in Σ.

In order to prove the main theorems there is need of several preliminary propositions, which we may state as lemmas.

8.1 Lemma 1. If $\phi(M)$ satisfies Condition N and equation (37) the function

$$\Phi(x) = \int_{y_1}^{y_2} \phi_x(x, \eta) \, d\, \eta$$

considered in any interval of x for which the line joining (x, y_1) to (x, y_2) lies in Σ, is a function of x of limited variation, and its total variation is $\leq T(\Sigma)+2\,N$, independently of the values of y_1 and y_2.*

In fact, we have the equation

$$F(C) = -\Phi(x_2)+\Phi(x_1) - \int_{x_1}^{x_2} \phi_y(\xi, y_2)\, d\,\xi + \int_{x_1}^{x_2} \phi_y(\xi, y_1)\, d\,\xi,$$

where by C is denoted the rectangle with vertices (x_1, y_1), (x_2, y_2) (x_1, y_2) (x_2, y_1); whence follows the result

$$|\,\Phi(x_2)-\Phi(x_1)\,| \leq T(c) + \int_{x_1}^{x_2} |\,\phi_y(\xi, y_2)\,| + |\,\phi_y(\xi, y_1)\,|\, d\,\xi.$$

Hence also

$$\Sigma_i\,|\,\Phi(x_{i+1})-\Phi(x_i) \leq T(\Sigma)+2\,N,$$

the points x_i representing an arbitrary separation of the total interval for x into partial intervals, with $x_i > x_{i+1}$. Hence the total variation of $\Phi(x)$ is at most equal to $T(\Sigma)+2\,N$, and the lemma is proved. The N is determined from the fact that for any straight line, $s < s_0$ the diameter of the region.

8.2 Lemma 2. Let $\phi(M)$ satisfy Condition N and equation (37). Let $v(M)$ denote a scalar point function continuous with its first partial derivatives, and let σ_0 denote a rectangular region with sides a and b contained in Σ, and s_0 its boundary. Then the equation

$$(38)\qquad \int_{\sigma_0} v(M)\, d\,F(s) = \int_{s_0} v\,\phi_n\, d\,s + \int_{\sigma_0}\left(\phi_x \frac{\partial\, v}{\partial\, x} + \phi_y \frac{\partial\, v}{\partial\, y}\right) d\,\sigma$$

is satisfied.

In order to prove the lemma we divide the sides a and b of the rectangle into m equal parts, and denote by ω_m the upper limit of the oscillations of the functions v, $\partial\, v/\partial\, x$, $\partial\, v/\partial\, y$ in any rectangle of sides a/m, b/m. We shall also use another such partition of the rectangle of index k with

* We say then that the function is of *uniformly* limited variation.

$k > m$. We specify the values of the coördinates corresponding to points of division as $x_0, x_1, \ldots, y_0, y_1, \ldots, (x_0, y_0)$ being for simplicity the origin, and proceed to write down the following expressions:

$$(\alpha') \quad \sum_{i=0, j=1}^{m-1, m} \left(\int_{y_{j-1}}^{y_j} \phi_x\,(x_i, y)\,d\,y \right) (v(x_{i+1}, y_j) - v(x_i, y_j)),$$

$$(\alpha'') \quad \sum_{i=1, j=0}^{m, m-1} \left(\int_{x_{i-1}}^{x_i} \phi_y(x, y_j)\,d\,x \right) (v(x_i, y_{j+1}) - v(x_i, y_j)),$$

$$(\beta') \quad \sum_{j=1}^{m} \int_0^a \left\{ \int_{y_{j-1}}^{y_j} \phi_x\,(x, y)\,d\,y \right\} d_x\,v(x, y_j),$$

$$(\beta'') \quad \sum_{i=1}^{m} \int_0^b \left\{ \int_{x_{i-1}}^{x_i} \phi_y(x, y)\,d\,y \right\} d_y\,v(x_i, y),$$

$$(\gamma') \quad \int_0^a d\,x \int_0^b \phi_x(x, y)\,\frac{\partial\,v(x, y)}{\partial\,x}\,d\,y,$$

$$(\gamma'') \quad \int_0^b d\,y \int_0^a \phi_y(x, y)\,\frac{\partial\,v(x, y)}{\partial\,y}\,d\,x,$$

$$(\delta) \quad \int_\sigma v(M)\,d\,F(s), \text{ which is equal to } \int_\sigma v(M)\,d \int_s \phi_n\,d\,s,$$

$$(\epsilon') \quad \sum_{i=1, j=1}^{m, m} v(x_i, y_j) \int_{y_{j-1}}^{y_j} \{\phi_x(x_i, y) - \phi_x(x_{i-1}, y)\}\,d\,y,$$

$$(\epsilon'') \quad \sum_{i=1, j=1}^{m, m} v(x_i, y_j) \int_{x_{i-1}}^{x_i} \{\phi_y(x, y_j) - \phi_y(x, y_{j-1})\}\,d\,x,$$

$$(\zeta) \quad \int_s v(M)\,\phi_n(M)\,d\,s,$$

$$(\eta') \quad \sum_{j=1}^{m-1} \left[v(a, y_j) \int_{y_{j-1}}^{y_j} \phi_x(a, y)\,d\,y - v(0, y_j) \int_{y_{j-1}}^{y_j} \phi_x(0, y)\,d\,y \right],$$

$$(\eta'') \quad \sum_{i=1}^{m-1} \left[v(x_i, b) \int_{x_{i-1}}^{x_i} \phi_y(x, b)\,d\,x - v(x_i, 0) \int_{x_{i-1}}^{x_i} \phi_y(x, 0)\,d\,x \right],$$

N A quantity $> \int |\,\phi_n\,|\,d\,s$ for any horizontal or vertical line in σ_0.

(α''') The expression (α') with $k>m$ substituted for m, in the summation with respect to the index i.

(α^{IV}) The expression (α'') with $k>m$ substituted for m, in the summation with respect to the index j.

We have the fact

$$|(\delta)+(\epsilon'')+(\epsilon')| \leq \omega_m T(\sigma_0) \tag{39}$$

from (5'') Part I. Also, by direct inspection,

$$\begin{aligned}(\epsilon') &= -(\alpha')+(\eta') \\ (\epsilon'') &= -(\alpha'')+(\eta'').\end{aligned} \tag{40}$$

Moreover

$$\begin{aligned}(\zeta) &= -\int_s v(M)\,\phi_x(M)\,d\,y+\int v(M)\,\phi_y(M)\,d\,x \\ &= -\left\{\int_0^b v(a,y)\,\phi_x(a,y)\,d\,y-\int_0^b v(0,y)\,\phi_x(0,y)\,d\,y\right\}+ \\ &\quad -\left\{\int_0^a v(x,b)\,\phi_y(x,b)\,d\,x-\int_0^a v(x,0)\,\phi_y(x,0)\,d\,x\right\}\end{aligned}$$

and therefore

$$|(\zeta)+(\eta')+(\eta'')| < \omega_m(4\,N). \tag{41}$$

By means of (39), (40) and (41) it follows that

$$\begin{aligned}|(\delta)-(\zeta)-(\alpha')-(\alpha'')| &= |(\delta)+(\epsilon')+(\epsilon'')-(\zeta) \\ &\quad -(\eta')-(\eta'')| \leq \omega_m\{4\,N+T(\sigma_0)\}.\end{aligned} \tag{42}$$

This equation may be regarded as half of the way towards the desired transformation, and we turn now to the expressions (β).

We have from Lemma 1,

$$\begin{aligned}|(\beta')-(\alpha''')| &\leq m\,\omega_k\{T(\sigma_0)+2\,N\} \\ |(\beta'')-(\alpha^{(\text{IV})})| &\leq m\,\omega_k\{T(\sigma_0)+2\,N\}.\end{aligned}$$

But from (42) we have

$$|(\delta)-(\zeta)-(\alpha''')-(\alpha^{\text{IV}})| \leq \omega_m\{4\,N+T(\sigma_0)\}$$

and therefore

$$\begin{aligned}|(\delta)-(\zeta)-(\beta')-(\beta'')| &\leq \omega_m\{4\,N+T(\sigma_0)\} \\ &\quad +2\,\omega_k\,m\{2\,N+T(\sigma_0)\}.\end{aligned} \tag{43}$$

Now

$$(\beta') = \sum_{j=1}^{m}\int_0^a\left\{\int_{y_{j-1}}^{y_j}\phi_x(x, y)\,d\,y\right\}\frac{\partial\, v(x, y_j)}{\partial\, x}\,d\,x,$$

so that

$$|\,(\beta')-(\gamma')\,| \leq \omega_m\, N\, a$$

$$|\,(\beta'')-(\gamma'')\,| \leq \omega_m\, N\, b.$$

Whence, by means of (42),

(44) $$|\,(\delta)-(\zeta)-(\gamma')-(\gamma'')\,| \leq \omega_m\{(4+a+b)\,N+T(\sigma_0)\} + 2\,\omega_k\, m\{2\,N+T(\sigma_0)\}.$$

The left-hand member of (44), which is independent of m, k can nevertheless be made $<\epsilon$ where ϵ is given arbitrarily in advance. In fact we choose first m so that

$$\omega_m < \frac{1}{(4+a+b)\,N+T(\sigma_0)}\,\frac{\epsilon}{2},$$

and then k so that

$$\omega_k < \frac{1}{2\,m\{2\,N+T(\sigma_0)\}}\,\frac{\epsilon}{2}.$$

Hence, finally,

(44′) $$(\delta) = (\zeta)+(\gamma')+(\gamma''),$$

and we have the identity (38), the passage from the iterated integral of (44′) to the double integral of (38) being immediate. This proves the lemma.

9. General Theorems. The identity (38) may now be extended to any curve or class Γ. Consider in fact a grating, the sides of any of the squares of which are $\leq\delta$ in length, and denote by σ_δ the region composed of all the squares which contain within or on their boundaries any of the interior or boundary points of σ. By s_δ denote the polygonal boundary of σ_δ. By Λ_δ denote the region $\sigma_\delta-\sigma$, and by λ_δ its boundary; the region Λ_δ is connected, and the total length of all its squares counted separately is not more then 12 times s. Assume that Condition N is verified.

9.1. Equation (38) applies to any square of the mesh and therefore to the region obtained by adding them all together and remains true as we let δ approach zero, even if at times the region σ_δ is not simply connected, and s_δ fails to consist of a single piece. Further, let us write

$$\int_\sigma v\, d\, F(s) = \int_s v\,\phi_n\, d\, s + \int_\sigma \left(\phi_x \frac{\partial v}{\partial x} + \phi_y \frac{\partial v}{\partial y}\right) d\,\sigma + H,$$

so that

$$\int_{\Lambda_\delta} v\, d\, F(s) = \int_{\lambda_\delta} v\,\phi_n\, d\, s + \int_{\Lambda_\delta} \left(\phi_x \frac{\partial v}{\partial x} + \phi_y \frac{\partial v}{\partial y}\right) d\,\sigma - H.$$

If we denote the several truncated squares that constitute the region Λ_δ by $\Lambda_\delta{}^i$ and by $\lambda_\delta{}^i$ its boundary and within any such subregion take a value of v, which we may denote by v_i, we shall have the relation

$$\left| \int_{\lambda_\delta} v\,\phi_n\, d\, s - \Sigma_i\, v_i \int_{\lambda_\delta{}^i} \phi_n\, d\, s \right| < \omega_\delta\, \Sigma_i \int_{\lambda_\delta{}^i} |\,\phi_n\,|\, d\, s < \omega_\delta\, k\, N,$$

where ω_δ is the upper limit of the oscillation of v in a square of side δ, and where N, k are finite and independent of δ. We have also the relation

$$\left| \int_{\Lambda_\delta} v\, d\, F(s) - \Sigma_i\, v_i\, F(\lambda_\delta{}^i) \right| < \omega_\delta\, T(\Lambda_\delta).$$

Moreover we have the identity

$$\Sigma_i\, v_i\, F(\lambda_\delta{}^i) = \Sigma_i\, v_i \int_{\lambda_\delta{}^i} \phi_n\, d\, s,$$

from the definition of the vector point function ϕ.

The quantity $T(\Lambda_\delta)$ does not necessarily approach 0 with δ; but ω_δ does, on account of the continuity of v. Hence

$$\lim_{\delta=0} \left[\int_{\Lambda_\delta} v\, d\, F(s) - \int_{\lambda_\delta} v\,\phi_n\, d\, s \right] = 0.$$

Also, evidently,

$$\lim_{\delta=0} \int_{\Lambda_\delta} \left(\phi_x \frac{\partial v}{\partial x} + \phi_y \frac{\partial v}{\partial y}\right) d\,\sigma = 0,$$

since Λ_δ approaches zero. Therefore the quantity H is zero, and the desired identity is proved. The theorem is as follows:

9.11 Theorem. *If ϕ is a vector point function which satisfies Condition N and equation* (38):

$$\int \phi_n \, d\, s = F(s)$$

for every s of Γ, and if $v(M)$ is a scalar point function continuous with its first partial derivatives, then the identity

$$\int_\sigma v\, d\, F(s) = \int_s v\, \phi_n\, d\, s + \int_\sigma \left(\frac{\partial\, v}{\partial\, x}\phi_x + \frac{\partial\, v}{\partial\, y}\phi_y\right) d\, \sigma \tag{45}$$

holds also for every s of Γ.

A vector point function which satisfies the equation(38) may be called a polarization vector for the distribution $F(s)$. The theorem just proved is thus an analysis of a function $v(M)$ in terms of a polarization vector for a certain distribution. It may easily be made much more general.

9.2 Theorem. *If ϕ is a vector point function which satisfies condition N and equation* (38)

$$\int \phi_n \, d\, s = F(s),$$

for every s of Γ, and if $v(M)$ is continuous and a potential function for its gradient vector $\nabla\, v$, the identity

$$\int_\sigma v\, d\, F(s) = \int_s v\, \phi_n\, d\, s + \int_\sigma (\nabla_x v\, \phi_x + \nabla_y v\, \phi_y)\, d\, \sigma \tag{46}$$

is valid for every s of Γ provided that one of the following additional hypotheses holds:

(α) $\nabla\, v(M)$ *is bounded, or*

(β) $\phi(M)$ *is bounded, or*

(γ) *the quantities $\{\nabla\, v(M)\}^2$ and $\{\phi(M)\}^2$ are summable superficially.*

In the present case, since $u(M)$ is continuous, we see as before from (22) and the accompanying theorem that $\partial\, v/\partial\, x$ and $\partial\, v/\partial\, y$ exist and are identical with $\nabla_x v$ and $\nabla_y v$ except possibly at the points of a set of superficial measure zero. Hence in (46) we may use instead of $\nabla_x v$, $\nabla_y v$, the corresponding partial derivatives according to the usual definition, or the corresponding generalized derivatives.

The theorem follows at once from the pseudo-polynomial approximation of Art. 7. In fact if we define

$$v^{(m)}(x, y) = \frac{1}{k_m^2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} v(x+\xi, y+\eta)\,(1-\xi^2)^m\,(1-\eta^2)^m\,d\xi\,d\eta,$$

and similarly the functions $v_1^{(m)}(x, y)$, $v_2^{(m)}(x, y)$ we have by (34) and (46)

$$\int_\sigma v^{(m)}\,dF(s) = \int_s v^{(m)}\phi_n\,ds + \int_\sigma (v_1^{(m)}\phi_x + v_2^{(m)}\phi_y)\,d\sigma,$$

whence if we let m become infinite, we have at once, by means of the theorem of Art. 7.1,

$$\lim_{m=\infty}\int_s v^{(m)}\phi_n\,ds = \int_s v\,\phi_n\,ds$$

$$\lim_{m=\infty}\int_\sigma (v_1^{(m)}\phi_x + v_2^{(m)}\phi_y)\,d\sigma = \int_\sigma \left(\frac{\partial v}{\partial x}\phi_x + \frac{\partial v}{\partial y}\phi_y\right)d\sigma,$$

and similarly, since $v^{(m)}$ approaches v uniformly in σ and since

$$\left|\int_\sigma (v - v^{(m)})\,dF(s)\right| < T(\sigma)\max|v - v^{(m)}|,$$

we have

$$\lim_{m=\infty}\int_\sigma v^{(m)}\,dF(s) = \int_\sigma v\,dF(s).$$

Hence the theorem is established.

If in this theorem we write

$$v(M) = h_m(M_0, M),$$

in which we define

$$\begin{aligned} M_0M &= r \\ h_m(M_0, M) &= \log 1/r, & r &\geq m \\ &= \log\frac{1}{m}, & r &< m, \end{aligned}$$

we get an interesting corollary by letting m approach zero.

9.21 Corollary. *Let $\phi(M)$ be a vector point function, which is a polarization vector for a distribution $F(s)$, and satisfies Condition N. Then, except for points M_0 which constitute at most a set of superficial measure zero, the identity*

$$(47)\quad \int_\sigma \log\frac{1}{r}\,d\,F(s) = \int_s \log\frac{1}{r}\,\phi_n\,d\,s - \int_\sigma \frac{1}{r^2}\{(x-x_0)\,\phi_x + (y-y_0)\,\phi_y\}\,d\,\sigma$$

is valid for a given curve s of Γ.

Let γ be the region interior to the circle m, and taking M_0 as a point not on s, let m be small enough so that the points of the circle are either all interior or all exterior points of σ. Since everywhere the function h is continuous, with first partial derivatives which remain finite, in numerical value $\leq 1/m$, equation (46) is valid and we have the equation

$$\int_\sigma h_m(M_0, M)\,d\,F(s) = \int_s \log\frac{1}{r}\,\phi_n\,d\,s - \int_{\sigma-\gamma} \frac{1}{r^2}\{(x-x_0)\,\phi_x + (y-y_0)\,\phi_y\}\,d\,\sigma.$$

If now we let m approach zero, as in Art. 3, we see that the two superficial integrals approach as limits the corresponding integrals of equation (47) for all points M_0 not on s, save for a further possible set of zero measure. Hence the Corollary is proved.*

9.3. If we restrict ourselves to solutions u, v of Poisson's equation we get a Green's theorem in some ways more general than that of Art. 9.2; in fact the functions u, v need not remain continuous or even finite, and their gradients need not satisfy Condition N. We consider therefore solutions of the equations

$$(48)\qquad \int \nabla_n u\,d\,s = F(s), \qquad \int \nabla v\,d\,s = G(s),$$

* Presented to the American Mathematical Society, September, 1919.

in which the discontinuities of $F(s)$ and $G(s)$ are merely of the first kind. We write these solutions in the forms

$$(48') \qquad u(M_1) = \frac{1}{2\pi}\int_\Sigma \log\frac{1}{r_1}\,df(e) + U(M_1),$$

$$v(M_2) = \frac{1}{2\pi}\int_\Sigma \log\frac{1}{r_2}\,dg(e) + V(M_2),$$

where $r_1 = M_1 M$ and $r_2 = M_2 M$, the functions $U(M_1)$, $V(M_2)$ being harmonic in the usual sense after their unnecessary discontinuities have been removed, in accordance with the theorem of Art. 6.3. We shall understand this to be done. The quantities $f(e)$ and $g(e)$ are the additive functions of point sets which by the method of Art. 4, correspond to $F(s)$ and $G(s)$ respectively.

9.31 Theorem. *Let $u(M)$, $v(M)$ be solutions of the equations* (48) *of the form* (48'). *A sufficient condition for the identity*

$$(49) \qquad \int_\sigma (\nabla_x u\,\nabla_x v + \nabla_y u\,\nabla_y v)\,d\sigma$$

$$= \int_\sigma u\,d\,G(s) - \int_s u\,\nabla_n v\,ds$$

$$= \int_\sigma v\,d\,F(s) - \int_s v\,\nabla_n u\,ds,$$

for all curves s of Γ, is the existence of the integral

$$(49') \qquad \int_\sigma d\,T_F(e)\int_\Sigma \log\frac{1}{r}\,d\,T_g(e).$$

For the proof of this theorem the $U(M_1)$ and $V(M_1)$ may be neglected, since, being completely harmonic, they enter into the demonstration in an entirely obvious manner. It is also sufficient to consider merely positive functionals $F(s)$, $f(e)$, $G(s)$, $g(e)$, since the given quantities may be expressed linearly in terms of those. For the purpose of the proof, then, we shall assume the equations

$$u(M_1) = \frac{1}{2\pi}\int_\Sigma \log\frac{1}{r_1}\,d\,f(e), \quad f(e) > 0,$$

$$v(M_2) = \frac{1}{2\pi}\int_{\Sigma} \log\frac{1}{r_2}\, d\, g(e), \quad g(e) > 0,$$

$$\int \nabla_n u \, ds = F(s), \quad F(s) > 0,$$

$$\int \nabla_n v \, ds = G(s), \quad G(s) > 0.$$

Let us define

$$h_m(M_1, M) = \log\frac{1}{r_1}, \quad r_1 \geq m$$
$$= \frac{1}{2} + \log\frac{1}{m} - \frac{r_1^2}{2m^2}, \quad r_1 < m,$$

a function of (x, y) or (x_1, y_1) which is continuous with its first partial derivatives. In fact:

$$\frac{d h_m}{d r} = -\frac{1}{r_1}, \quad r_1 \geq m$$
$$= -\frac{r_1}{m^2}, \quad r_1 < m.$$

Moreover for $\partial h_m/\partial n$ we have the inequality

$$\left|\frac{\partial h_m}{\partial n}\right| \leq \frac{|\cos r_1 n|}{r_1}.$$

We define now:

$$u_m(M_1) = \frac{1}{2\pi}\int_{\Sigma} h_m(M_1, M)\, df(e),$$

$$v_k(M_2) = \frac{1}{2\pi}\int_{\Sigma} h_k(M_2, M_1)\, d\, g(e^1),$$

functions which are finite and continuous, with finite and continuous first partial derivatives. But we may calculate the second derivatives of $h_m(M_1, M)$; in fact

$$\frac{\partial^2 h_m}{\partial x_1^2} + \frac{\partial^2 h_m}{\partial y_1^2} = 0, \qquad r_1 \geq m$$
$$= -\frac{2}{m^2}, \quad r_1 < m,$$

so that since

$$\int_{s_1} \frac{\partial\, u_m(M_1)}{\partial\, n_1}\, d\, s_1 = \frac{1}{2\,\pi} \int_{\Sigma} d\, f(e) \int_{s_1} \frac{\partial\, h_m(M_1,\, M)}{\partial\, n_1}\, d\, s_1$$

$$= \frac{1}{2\,\pi} \int_{\Sigma} d\, f(e) \int_{\sigma_1} (-\nabla_1^2\, h_m(M_1,\, M))\, d\, \sigma,$$

we have

$$\left| \int_{s_1} \frac{\partial\, u_m(M_1)}{\partial\, n_1}\, d\, s_1 \right| \leq \frac{1}{\pi\, m^2}\, \sigma_1\, f(\Sigma),$$

and we may write

$$\int_{s} \frac{\partial\, u_m(M)}{\partial\, n}\, d\, s = F_m(s)$$

where $F_m(s)$ is a positive additive (in fact, absolutely continuous) function of curves. A similar fact holds for the gradient of $v_k(M_2)$ which defines a functional $G_k(s)$.

The gradients of $u_m(M_1)$ and $v_k(M_2)$ being finite, satisfy Condition N and therefore, by the theorem of Art. 9.1 or 9.2, the identity (49) will apply to these functions. We may then let m and k separately approach zero, and see what result can be obtained.

9.311. If we consider the first part of (49) and let k become zero, the resulting identity is still valid. In fact we have

$$\lim_{k=0} \nabla\, v_k = \nabla\, v,$$

as a vector equation; moreover

$$|\, u_n| \leq \frac{1}{2\,\pi} \log \frac{1}{m} f(\Sigma),$$

$$\nabla_x\, u_m \leq \frac{f(\Sigma)}{2\,\pi\, m},$$

so that

$$|\, \nabla_x\, u_m\, \nabla_x\, v_k + \nabla_y\, u_m\, \nabla_y\, v_k\, | \leq \frac{f(\Sigma)}{2\,\pi\, m} \frac{1}{2\,\pi} \int_{\Sigma} \frac{1}{r}\, d\, g(e'),$$

and this latter expression as a function of M is integrable over Σ. Since, moreover, it is independent of k, it follows that

$$\lim_{k=0} \int_\sigma (\nabla_x u_m \nabla_n v_k + \nabla_y u_m \nabla_y v_k)\, d\sigma = \int_\sigma (\nabla_x u_m \nabla_x v + \nabla_y u_m \nabla_y v)\, d\sigma.$$

Similarly it may be deduced that

$$\lim_{k=0} \int_s u_m \nabla_n v_k\, ds = \int_s u_m \nabla_n v\, ds,$$

because

$$|u_m \nabla_n v_k| \leq \frac{1}{2\pi} \log \frac{1}{m} f(\Sigma) \cdot \frac{1}{2\pi} \int_\Sigma \frac{|\cos r n|}{r}\, d g(e'),$$

and this expression, which is independent of k, may be integrated around any curve of class Γ. The value of the integral is

$$\leq \frac{1}{4\pi} \Gamma \log \frac{1}{m} f(\Sigma)\, g(\Sigma).$$

It remains to consider the third term

$$\int_\sigma u_m\, d G_k(s).$$

In order to find the limit of this expression we must paraphrase a theorem proved by H. E. Bray for a Stieltjes integral in one dimension.[16]) We shall state it rather less generally than the direct extension, but nevertheless subject to exactly the same method of proof.

9.312 Lemma. Let $G(s)$, $G_k(s)$ be additive functions of curves of limited variation, the latter being of uniformly limited variation with respect to k: and let $\phi(M)$ be a continuous function of M. If for every s of Γ,

$$G(s) = \lim_{k=0} G_k(s),$$

[16]) H. E. Bray, "Elementary properties of the Stieltjes integral," Annals of Mathematics, vol. 20 (1918–19), pp. 177–186. See page 180. $G_k(s)$ is of uniformly limited variation with respect to k, if $T_k(s)$ remains finite irrespective of k.

then for every s of Γ

$$\int \phi(M)\, d\,G(s) = \lim_{k=0} \int \phi(M)\, d\,G_k(s).$$

But as we have already seen,

$$\lim_{k=0} \int \frac{\partial v_k}{\partial n}\, d\,s = \int \nabla_n v\, d\,s,$$

and also $G_k(s)$ is a positive function of curves, such that

$$T_k(s_1) = G_k(s_1) = \frac{1}{2\,\pi} \int_\Sigma d\,g(e) \int_{\sigma_1} (-\nabla_1{}^2\, h_m(M_1,\, M)\, d\,\sigma,$$
$$\leq \frac{1}{2\,\pi} \int_\Sigma d\,g(e) \int_{\sigma_m} \frac{2}{m^2}\, d\,\sigma,$$

where σ_m is a circle of radius m and center M. Hence

$$T_k(s_1) \leq \frac{1}{2\,\pi}\, (\pi\, m^2)\, \frac{2}{m^2}\, g(\Sigma)$$
$$\leq g(\Sigma),$$

and $G_k(s)$ is of uniformly limited variation with respect to k; and the conditions of the lemma are satisfied. Hence finally

$$\lim_{k=0} \int_\sigma u_m\, d\,G_k(s) = \int_\sigma u\, d\,G(s),$$

and the formula

$$(50) \qquad \int_\sigma (\nabla_x u_m\, \nabla_x v + \nabla_y u_m\, \nabla_y v)\, d\,\sigma = \int_\sigma u_m\, d\,G(s) - \int u_m\, \nabla_n v\, d\,s,$$

is valid.

9.313. We are now in a position to let m approach zero. We have for a point M_1,

$$|\, \nabla_x u_m\, \nabla_x v + \nabla_y u_m\, \nabla_y v\, | \leq \int_\Sigma d\,f(e) \int_\Sigma \frac{1}{r\,r'}\, d\,g(e')$$

independently of m, and we have to decide whether or not this is a summable function of M_1. We can, however, form directly

$$\int_{\sigma_1} \frac{d\,\sigma_1}{r\,r'},$$

choosing σ_1 as a circle of radius R sufficiently large. If we denote by ρ the distance $M M'$ between the extremities of $r' = M' M$ and $r = M_1 M$ we find that this integral is $< C_1 + C_2 \log 1/\rho$, with C_1 and C_2 definite constants. Hence our expression does represent a summable function of M_1 provided that

$$\int d f(e) \int \log \frac{1}{\rho} \, d g(e')$$

is convergent. And this is the hypothesis of our theorem if $f(e)$, $g(e)$ are ≥ 0 for every e.

The same condition is sufficient in order that

$$\lim_{m=0} \int_\sigma u_m \, d G(s) = \int_\sigma u \, d G(s).$$

In fact $| u_m$ is $\leq \left| \int_\Sigma \log \frac{1}{r} \, d f(e) \right.$, irrespective of m.

The remaining integral is a curvilinear integral. The functions u_m form an increasing sequence, and therefore,

$$\lim_{m=0} \int_s u_m \, \nabla_n v \, d s = \int u \, \nabla_n v \, d s,$$

whenever the latter integral is convergent in the Lebesgue sense. But we can investigate the integral even more closely.

Let s be any convex curve of Γ, and introduce polar coördinates with M' as pole. The relation

$$\int_{s_1} \log \frac{1}{r} \frac{|\cos n \, r'|}{r'} \, d s' = \int_{s_1} \log \frac{1}{r} \, | d \theta' |$$

is immediate. The convex curve may, however, be divided into two parts in each of which the angle θ' changes monotonically by $\leq \pi$. If we consider one such part s' and let σ' denote the sector of which it forms the base, and M' the vertex, we shall have in the case $M \neq M'$, that

$$\int_{s'} \log \frac{1}{r} \, | d \theta' | = \int \frac{1}{r} \left| \frac{d r}{d r'} \right|_{\theta' \text{ const.}} d r' \, d \theta'$$

$$= \int_{\sigma'} \frac{1}{r \, r'} \left| \frac{d r}{d r'} \right|_{\theta' \text{ const.}} d \sigma' \leq \int_{\sigma'} \frac{1}{r \, r'} \, d \sigma'.$$

But this integral has already been treated. And therefore for any convex curve s_1, with $M \neq M'$ we have

$$\int_{s_1} \log \frac{1}{r} \frac{|\cos n\, r'|}{r'} \, d\, s \leq C_1 + C_2 \log \rho$$

with proper values of C_1 and C_2.

The same relation applies if s_1 is any curve which may be decomposed into a finite number of convex curves, or more generally, if s_1 is any curve of class Γ, since what is important for the proof is that $\int_{s_1} | \mathrm{d}\, \theta' |$ remains finite.

We see then that if $\int d\, f(e) \int \log 1/\rho \; d\; g(e')$ remains finite, the quantity

$$\int |u(M_1)\, \nabla_n v(M_1)|\, d\, s_1 \leq \int_\Sigma d f(e) \int_\Sigma d\, g(e') \int_{s_1} \log \frac{1}{r} \frac{|\cos n\, r'|}{r'} \, d\, s_1$$

converges, and we are justified in the passage to the limit. And thus the first of the equation (49) is demonstrated. The second results merely from the interchange of u and v.

If now we replace in $f(e)$ and $g(e)$ their original interpretations as arbitrary additive functions of point sets, no longer necessarily positive, we shall have to replace them in the condition of convergence by their total variations; and in this way we arrive at the hypothesis of the theorem of Art. 9.31 as a sufficient condition.

10. Frontier integrals. The curvilinear integrals with which we have been dealing in the last section are examples of functionals which depend on curves, uniquely defined for all curves of class Γ. When these integrals are functions of curves of limited variation, their associated functions of point sets are uniquely determined for all point sets measurable in the sense of Borel, according to the correspondence established in Art. 4. As has been noted by P. J. Daniell*

* See [14]).

a generalized definition of such integrals may be given in terms of this result, to apply to the complete boundary of any set measurable in the Borel sense. We shall designate such quantities by the term "frontier integrals," since according as the boundary is regarded as the frontier of one set or another, the integral may have different values. For the complete definition of a frontier integral it is sufficient, therefore, as we have seen, to know its value for all rectangles.

Hence it is unnecessary for the consideration of frontier integrals to assume Condition N. It is sufficient if the vector ϕ is summable over Σ and if the integral of its normal component on any vertical or horizontal line remains finite, $<N$.

If the curvilinear integrals on s are replaced throughout by the frontier integrals defined for the open sets σ, and the functions of curves $F(s)$, $G(s)$ replaced throughout by the corresponding functions of point sets $f(e)$, $g(e)$, the theorems of Art. 9.2, 9.3 and their Corollary hold, and apply to all sets measurable in the Borel sense. It is unnecessary to state them all in this new form, but one may be written down in order to exemplify the notation. Following Daniell, the frontier of the set E is denoted by $B(E)$.

10.1. Theorem. *If for every rectangle, we have*

$$\int_{B(E)} \nabla_n u \, d s = f(E) \tag{52}$$

$$\int_{B(E)} \nabla_n v \, d s = g(E)$$

where $f(e)$ and $g(e)$ are additive functions of point sets, the solutions may be written in the form

$$u(M_1) = \frac{1}{2\pi} \int_\Sigma \log \frac{1}{r} \, d f(e) + U(M_1) \tag{52'}$$

$$v(M_1) = \frac{1}{2\pi} \int_\Sigma \log \frac{1}{r} \, d g(e) + V(M_1)$$

in which $U(M_1)$ and $V(M_1)$ are harmonic functions, and in this form, for any set E measurable in the Borel sense the following identity is valid:

$$(52'') \quad \int_E (\nabla_x u \, \nabla_x v + \nabla_y u \, \nabla_y v) \, d E$$

$$= \int_E u \, d \, g(e) - \int_{B(E)} u \, \nabla_n v \, d s$$

$$= \int_E v \, d f(e) - \int_{B(E)} v \, \nabla_n u \, d s$$

provided the integral

$$\int_\Sigma d \, t_f(e) \int_\Sigma \log \frac{1}{r} d \, t_g(e')$$

is convergent.

11. Further Corollaries. We may close this chapter by calling to mind the usual corollaries of our theorems. Thus we obtain useful special cases by putting $v \equiv u$ and $v \equiv \log \frac{1}{r}$ where $r = M_0 M$, the distance from a fixed point.

11.1. Corollary. *Let $u(M)$ be a solution of*

$$\int_s \nabla_n u \, d s = F(s),$$

in the form

$$u(M_1) = \frac{1}{2\pi} \int_\Sigma \log \frac{1}{r} d f(e) + U(M_1),$$

with $U(M_1)$ harmonic. The equation

$$(53) \quad \int_\sigma (\nabla u)^2 \, d \sigma = \int_\sigma u \, d \, F(s) - \int_s u \, \nabla_n u \, d s$$

holds, provided the quantity

$$\int_\Sigma d \, t(e) \int_\Sigma \log \frac{1}{r} d \, t(e)$$

exists: Also

$$(53') \quad \int_E (\nabla u)^2 \, d \sigma = \int_E u \, d f(e) - \int_{B(E)} u \, \nabla_n u \, d s.$$

11.2. Corollary 3. *With the same hypotheses as Corollary 2 the equations*

(54) $2\pi\, u(M_0)$

$$= \int_s \left\{\frac{\cos n\, r}{r} u + \log\frac{1}{r} \nabla_n u\right\} d\, s - \int_\sigma \log\frac{1}{r}\, d\, F(s)$$

$$= \int_s \frac{\cos n\, r}{r} u\, d\, s + \int_\sigma \frac{1}{r}\left\{\nabla_x u \cos x\, r + \nabla_y u \cos y\, r\right\} d\, \sigma$$

$$= \int_{B(E)} \left\{\frac{\cos n\, r}{r} u + \log\frac{1}{r} \nabla_n u\right\} d\, s - \int_E \log\frac{1}{r}\, d f(e)$$

$$= \int_{B(E)} \left\{\frac{\cos n\, r}{r} u\right\} d\, s + \int_E \frac{1}{r}\{\nabla_x u \cos x\, r + \nabla_y u \cos y\, r\}\, d\, E,$$

hold for all points M_0 except possibly those which form a certain set of superficial measure zero, this set being independent of σ and E.

3. *The Dirichlet Problem*

12. The Green's function. An interesting application of these general ideas is to the Dirichlet problem; to determine a harmonic function throughout an open region, by means of assigned frontier values. For this purpose we shall consider an arbitrary region as defined by Osgood [17]) but with a boundary consisting of more than one point so that it possesses a Green's function,* and for simplicity restricted to a finite domain.[18]) The following requirements are then sufficient:

(i) The region T is *open, i.e.,* every point is an interior point.

(ii) The region T is finitely extended.

(iii) The region T is finitely connected.

[17]) Osgood, "Lehrbuch der Funktionentheorie," Leipzig (1912), p. 151.

* See [17]), p. 630.

[18]) Plemelj, "Potential theoretische Untersuchungen," Leipzig (1911), takes the infinite domain to correspond to the infinite point of the complex variable, and imposes suitable conditions on the potential so that the point at infinity is not a special point.

Such a region may be regarded as consisting of the points of a denumerable aggregate of non-overlapping rectangular regions.

The Green's function $g(M_0, M)$ may be written as

$$g(M_0, M) = \log \frac{1}{r} + g'(M_0, M), \tag{55}$$

the distance M_0M being r, with M_0 an internal point, and $g'(M_0, M)$ completely harmonic for M in T. For M on the frontier of T, $g(M_0, M) = 0$. Conjugate to $g(M_0, M)$ is the function $h(M_0, M)$ also harmonic in T, but of course multiple valued. The two functions (g, h) for our purposes form a convenient set of polar curvilinear coördinates for the region T.

13. The problem for the circle. In order to separate the difficulties, we take up in detail first the case of a circle of radius a, and limit ourselves to functions $u(M)$ which are harmonic at all interior points of the circle; the first partial derivatives of $u(M)$ together with their squares are assumed to be summable superficially in the Lebesgue sense over the circle.

Consider now a sub-region σ which does not include the center 0 of the circle, with a boundary s. Since $\frac{1}{r}\frac{\partial u}{\partial r}$ is summable (L) over such a region, by our hypothesis, the quantity

$$\int_s u \, d\theta = \int_\sigma \frac{1}{r}\frac{\partial u}{\partial r}\, d\sigma$$

represents an absolutely continuous function of point sets σ (or curves s). Hence if we let $\phi_m(\xi)$ be a continuous function of ξ, $\theta_1 \leq \xi \leq \theta$ such that

$$\lim_{m=\infty} \phi_m(\xi) = a,$$

and integrate along the curve $r=\phi_m(\theta)$ from θ_1 to θ we define a function

$$H(\theta)=\lim_{m=\infty}\int_{\theta_1}^{\theta} u\,d\theta.$$

This function, moreover, is absolutely continuous as a function of θ. In fact, referring to the diagram,

$$H(\theta_2)-H(\theta_1)$$

$$=\int_A^B u\,d\theta+\int_{\bar\sigma} r\frac{\partial u}{\partial r}\,d\sigma,$$

in which expression $A\,B$ is the arc of a circle of radius $R\neq 0$. Since on $A\,B$, u is harmonic, it remains finite, $<M$, and therefore

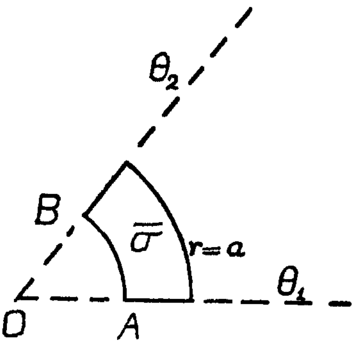

Figure 1.

$$|H(\theta_2)-H(\theta_1)|\leq M\,|\theta_2-\theta_1|+\int_{\bar\sigma}\left|\frac{1}{r}\frac{\partial u}{\partial r}\right|d\sigma,$$

and the total variation of $H(\theta)$ from 0 to 2π is

$$\leq M\,2\pi+\int\left|\frac{1}{r}\frac{\partial u}{\partial r}\right|d\sigma,$$

the region of integration in the last integral being the ring of inner radius R and outer radius a.

But

$$M\,|\theta_2-\theta_1|\quad\text{and}\quad\int_\sigma\left|\frac{1}{r}\frac{\partial u}{\partial r}\right|d\sigma$$

define absolutely continuous functions of θ_2, and therefore the same result applies to $H(\theta_2)-H(\theta_1)$.

Since $H(\theta)$ is an absolutely continuous function of θ, it defines a function

$$\bar u(M)=\frac{d}{d\theta}H(\theta),$$

such that

$$H(\theta)=\int_{\theta_1}^{\theta_2}\bar u(M)\,d\theta.$$

It turns out that this boundary function $\bar u(M)$, now defined, is characteristic of the harmonic function $u(M)$.

13.1. In order to investigate this statement, consider an arbitrary point M_0, interior to the circle a, and through it draw the family of circles which cut the boundary $r=a$ normally. These circles may be individualized by a parameter ψ used to denote the angle made with a fixed direction by the tangent at M_0 to the circle of the family. These circles with their orthogonal trajectories $\chi=$const. determine a set of curvilinear coördinates, which correspond precisely to the level lines of conjugate function and Green's function respectively.

The function

$$K(\psi)=\lim_{m=\infty}\int_{\psi_1}^{\psi} u\, d\psi,$$

where we integrate along a curve $\chi=\chi_m(\psi')$, continuous $\psi_1 \leq \psi' \leq \psi$, and let the curve approach the outside boundary as m becomes infinite, defines as in the special case just treated when M_0 was the origin, a uniformly continuous function of ψ; and this is the integral of its derivative $\bar{v}(M)$ with respect to ψ. *This function $\bar{v}(M)$ is identical with the $\bar{u}(M)$ "almost everywhere" on the boundary.*

This result may now be verified. Take an arc of a circle with the center O to serve both as $r=\phi(\theta)$ and $\chi=\chi(\psi)$, the radius r_m of the arc being $>OM_0$. Let θ_1 and ψ_1 correspond to one point on the circle $r=a$, and θ_2 and ψ_2 to a second point. Let A, B, C, D be the points on $r=r_m$ determined by ψ_1, ψ_2, θ_1, θ_2 respectively. When necessary to avoid confusion let the subscript m indicate a value referring to r_m and the subscript a refer similarly to r.

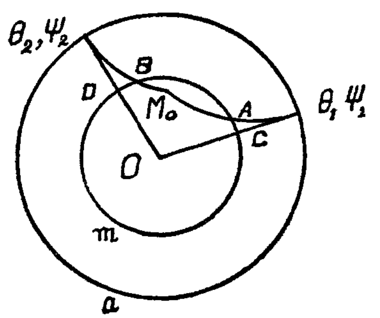

Figure 2.

We have

$$\int_{\theta_1}^{\theta_2} u_m\, d\theta = \int_{\psi_1}^{\psi_2} u_m \frac{\partial \theta}{\partial \psi}\, d\psi + \int_B^D u_m\, d\theta - \int_A^C u_m\, d\theta,$$

in which it can be easily shown that the last two integrals of the second member approach zero as m becomes infinite. In fact:

$$\int_B^D u_m\, d\theta = \int_{\theta_B}^{\theta_2} u_{m_1}\, d\theta + \int_{\sigma_m} \frac{1}{r}\frac{\partial u}{\partial r}\, d\sigma,$$

in which m_1 is a fixed value of m, and σ_m is the region bounded by $\theta = \theta_B$, $\theta = \theta_2$ and $r = r_{m_1}$ and $r = r_m$. This region approaches zero as m becomes infinite, since θ_B approaches θ_2; moreover the integral extended over it approaches zero, since the integral over the whole circle outside $r = m_1$ is convergent. The integral extended over the curve $r = r_{m_1}$ also approaches zero, and therefore $\int_B^D u_m\, d\theta$ approaches zero. Similar reasoning holds for $\int_A^C u_m\, d\theta$.

We have then

$$\lim_{m=\infty} \int_{\theta_1}^{\theta_2} u_m\, d\theta = \lim_{m=\infty} \int_{\psi_1}^{\psi_2} u_m \frac{d\theta}{d\psi}\, d\psi,$$

or

$$\int_{\theta_1}^{\theta_2} \bar{u}\, d\theta = \lim_{m=\infty} \int_{\psi_1}^{\psi_2} u_m \frac{d\theta}{d\psi}\, d\psi.$$

But $\int_{\psi_1}^{\psi_2} \bar{v} \frac{d\theta}{d\psi}\, d\psi$ has a definite value on $r = a$, and also,

$$\int_{\psi_1}^{\psi_2} \bar{v} \frac{d\theta}{d\psi}\, d\psi = \left(\int_{\psi_1}^{\psi_2} \bar{v}\, d\psi\right) \left[\left(\frac{d\theta}{d\psi}\right)_a\right]_{\psi_1}^{\psi_2} - \int_{\psi_1}^{\psi_2} \left(\int_{\psi_1}^{\psi} \bar{v}\, d\psi\right) \left(\frac{d^2\theta}{d\psi^2}\right)_{r=a} d\psi,$$

$$\int_{\psi_1}^{\psi_2} u_m \frac{d\theta}{d\psi}\, d\psi = \left(\int_{\psi_1}^{\psi_2} u_m\, d\psi\right) \left[\left(\frac{d\theta}{d\psi}\right)_m\right]_{\psi_1}^{\psi_2} - \int_{\psi_1}^{\psi_2} \left(\int_{\psi_1}^{\psi} u_m\, d\psi\right) \left(\frac{d^2\theta}{d\psi^2}\right)_m d\psi,$$

so that
$$\lim_{m=\infty}\int_{\psi_1}^{\psi_2} u_m \frac{d\theta}{d\psi}\,d\psi = \int_{\psi_1}^{\psi_2} \bar{v}\frac{d\theta}{d\psi}\,d\psi;$$
for the quantities
$$\frac{d\theta}{d\psi},\quad \frac{d^2\theta}{d\psi_2},\quad \int_{\psi_1}^{\psi} u_m\,d\psi,$$
are continuous functions of ψ and remain finite as m becomes infinite.

We may now write:
$$\int_{\psi_1}^{\psi_2} \bar{v}\frac{d\theta}{d\psi}\,d\psi = \int_{\theta_1}^{\theta_2} \bar{v}\,d\theta,$$
and therefore
$$\int_{\theta_1}^{\theta_2} \bar{u}\,d\theta = \int_{\theta_1}^{2} \bar{v}\,d\theta,$$
a relation which holds for any value of θ_2. Hence $\bar{v}(M) \equiv \bar{u}(M)$ on the circle $r=a$ except at most for points of a set of linear measure zero. The function $\bar{u}(M)$ may therefore be regarded as characteristic of the harmonic function $u(M)$.

13.2. Conversely *given $\bar{u}(M)$ summable (L) there is not more than one function $u(M)$ harmonic within the circle, whose first derivatives are summable with their squares, such that the integral along an arbitrary curve approaches the integral of $\bar{u}(M)$ as the curve approaches the boundary of the circle.* For $u(M)$ being harmonic within the circle, without other conditions, satisfies the mean value theorem for curves $\chi =$ const.; that is
$$u(M_0) = \frac{1}{2\pi}\int_0^{2\pi} u_m(M)\,d\psi,$$
and as m becomes infinite,
$$u(M_0) = \frac{1}{2\pi}\int_0^{2\pi} \bar{u}(M)\,d\psi,$$
and is therefore uniquely determined.

13.3. So far we have not drawn any conclusion about the behavior of $u(M_0)$ itself as M_0 approaches a point M of the boundary; and from the mere behavior of the integral, we cannot tell that there is any limiting value of $u(M_0)$. We have, however, a theorem which we shall now proceed to prove, which depends, however, upon the following lemma.

13.31 Lemma. Let $F(x, y)$ be continuous in x, y in the region $a \leq x \leq b$, $0 < y \leq y_0$; let $f(x)$ be summable (L) in the interval $a \leq x \leq b$, and let $\phi(x)$ be a positive, continuous function, $a \leq x \leq b$ which defines a curve l across the given region.

If for all $\phi(x) \leq \delta$, we have

$$\lim_{\delta=0} \int_a^b | F(M_e) - f(x) | \, dx = 0,$$

then we have

$$\lim_{y=0} F(x, y) = f(x)$$

almost everywhere in the interval $a\ b$.

Let us first prove the theorem in the case that $f(x)$ itself is continuous $a \leq x \leq b$. Consider the set of points on the x-axis such that for points of the set the oscillation of $| F(x, y) - f(x) |$ does not approach 0 with y but remains $> \eta$. This set is measurable for every value of η, and if the theorem is false, can be chosen (small enough) so that the measure of the corresponding set E is not zero, but, say $\geq e$.

Let us suppose now that δ is chosen small enough so that

$$\int_a^b | F(M_l) - f(x) | \, dx < \frac{1}{2} e \eta$$

for every $| \phi(x) | \leq \delta$ and consider specially the set of points in the region $y < \delta$ for which $| F(x, y) - f(x) | > \eta$.

Since $f(x)$ is continuous this is an open set, and therefore is equivalent to a denumerable infinity of rectangles, with sides parallel to the x and y axes. Let these rectangles be numbered in order of decreasing area, and in the case of two or more rectangles of the same area, let the rectangle of largest base precede.

Let $y = y_1$ be any line which cuts the rectangle (1) and between values $x = x_1'$ and $x = x_1''$. If the rectangle (2) projects beyond this, let $y=y_2'$ be a line that cuts (2) and $x_2'\ x_2''$ the portion of this line not already included in $x_1'\ x''_1$. Proceeding in this way, we can in a finite number n of steps create a set of non-overlapping intervals x which together constitute a length more than any given proper fraction — say one half — of E. For suppose the intervals approach a limiting measure, $<\frac{1}{2}\, e$. We could then find a rectangle of finite size whose projection on the rest of E would not be of zero measure, and we should not have included this rectangle in our sequence.

Hence if we define the step function

$$\begin{aligned} \bar{\phi}(x) &= y_1, & x_1' &< x < x_1'', \\ &= y_2, & x_2' &< x < x_2'', \\ &\cdots\cdots & &\cdots\cdots\cdots \\ &= y_n, & x_n' &< x < x_n'', \\ &= \delta, & &\text{otherwise,} \end{aligned}$$

we shall have

$$\int_a^b \mid F(M_{\bar{\phi}}) - f(x) \mid dx > \frac{1}{2}\, e\,\eta,$$

and since $F(x, y)$ is continuous, we can replace $\bar{\phi}$ by a continuous ϕ such that $\mid F(M_{\bar{\phi}}) - F(M_\phi) \mid$ is uniformly as small as we please. Hence we can find $\phi(x)$, $0 < \phi(x) \leq \delta$, and continuous, so that

$$\int_a^b \mid F(M_e) - f(x) \mid dx > \frac{1}{2}\, e\,\eta,$$

which is a contradiction.

Suppose now that $f(x)$ is no longer necessarily continuous, but summable (L). We can find a continuous function $g(x)$ which outside a point set G of given measure γ, arbitrarily small differs from $f(x)$ by less say than $\eta/3$. Let E_f be the set of points for which

$$\lim_{y=0} \text{Osc.} \mid F(x, y) - f(x) \mid > \frac{4}{3}\, \eta,$$

Insert in THE RICE INSTITUTE PAMPHLET,
VOL. VII, No. 4, between pages 318 and 319.

ERRATA CORRIGE

1. In the theorems of Arts. 13.4 and 15.1, pages 319 and 327 respectively, delete the words: *and with first partial derivatives which, with their squares, are summable.* This parenthetical statement is not true, as is shown by an example given by Hadamard (*Bulletin de la Societé Mathématique de France*, Vol. 34, 1906). The mistake arises in Art. 13.41, which should therefore be omitted.

The theorem under discussion in its amended form has been proved, however, for the circle, $f(M)$ being an arbitrary function, bounded and summable (L), by Fatou (*Acta Mathematica*, Vol. 30, 1907), and with this as a basis the amended theorem for the general T region follows as before.

2. On page 320, line 7, replace *bounded* by *summable.*

On page 322, lines 9 and 12, replace *measurable* (B) by *of class* T_2, in line 16 replace *measurable* (B) by *of previous class*, and in line 13 replace *a finite number of steps* by *two stages.*

and E_g the portion of it which does not contain points of G, of measure e_g. Take now δ small enough so that

$$\int_a^b | F(M_l) - f(x) | \, dx < \frac{1}{6} e_g \eta,$$

and therefore so that

$$\int_a^b | F(M_l) - g(x) | \, dx < \frac{1}{2} e_g \eta,$$

But for every x of E_g

$$\begin{aligned}\lim_{y=0} \text{Osc.} \, | F(x, y) - g(x) | &> \frac{4}{3} \eta - \frac{1}{3} \eta \\ &> \eta,\end{aligned}$$

and since $g(x)$ is continuous we can get a contradiction between this and the previous inequality, in the same way as before.

Let us now state the fundamental theorem for the circle.

13.4. *Given $f(M)$ on the circle $r = a$ summable in the Lebesgue sense, if $f(M)$ is bounded, $< N$, there is one and only one bounded function $u(M)$, harmonic inside the circle, and with first partial derivatives which, with their squares, are summable, which takes on the values $f(M)$ almost everywhere on the boundary. This solution is given by Poisson's integral.*

We say that $u(M)$ takes on the values of $f(M)$ *almost everywhere* if given M_1, an interior point, and the circles $\psi =$const. through M_1 as pole; we have, letting M' move along such a curve,

$$\lim_{M' = M} u(M') = f(M)$$

for *almost all* the curves $\psi =$const. If this relation is satisfied when we choose one pole M_1, we shall see that in the case of our theorem it is satisfied for any other; and therefore a sufficient test will be that for *almost all* radii,

$$\lim_{r=a} u(M') = \bar{u}(M)$$

as M' moves along the radius.

We see at once that there can be only one function $u(M)$ which satisfies the conditions of the theorem, for if $u(M)$ is bounded and satisfies the above relation for one pole M_1 we have (with the coördinates of Art. 13.1)

$$\lim_{x=0} \int_{\psi_1}^{\psi_2} u\, d\psi = \int_{\psi_1}^{\psi_2} f\, d\psi. \tag{56}$$

But if the first partial derivatives of a harmonic function are bounded with their squares, over the circle, it has a characteristic function $\bar{u}(M)$ on the boundary, according to the theorems of Art. 13.1 and it is uniquely determined by that characteristic function. By (56), however, we must have

$$\bar{u}(M) = f(M),$$

almost everywhere.

13.41. The function given by Poisson's integral is obviously in modulus $\leq N$ if $|f(M)|$ is $\leq N$. We shall show that its first partial derivatives are summable with their squares, and we shall do this by means of a calculation of $|\partial u/\partial n|$ on an arbitrary circle. In fact, if we integrate over a circle of radius r,

$$\int_\sigma \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} d\sigma = -\int_s u \frac{\partial u}{\partial n}\, ds$$
$$\leq N a \int_0^{2\pi} \left|\frac{\partial u}{\partial r}\right| d\theta,$$

and it is sufficient to show that the last integral remains finite as r approaches a.

We may write Poisson's integral in the forms

$$\begin{aligned} u(M') &= u(x', y') \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(M)\, \frac{a^2 - r^2}{a^2 - 2ar\cos(\phi - \theta) + r^2}\, d\phi \\ &= \frac{1}{\pi}\int_0^{2\pi} f(M) \left\{\frac{d}{d\phi} \arctan \frac{y - y'}{x - x'} - \frac{1}{2}\right\} d\phi \\ &= -\frac{1}{\pi}\int_0^{2\pi} f(M) \left\{\frac{\partial}{\partial \theta} \arctan \frac{y - y'}{x - x'} - \frac{1}{2}\right\} d\phi, \end{aligned} \tag{57}$$

in which (a, ϕ) are the polar coördinates of M and (r, θ) the polar coördinates of M'.

Now we have

$$\frac{\partial u}{\partial r}=\frac{a}{\pi}\int_0^{2\pi}\frac{\cos(\phi-\theta)\,(a^2+r^2)-2\,a\,r}{\{a^2-2\,a\,r\cos(\phi-\theta)+r^2\}^2}f(M)d\,\phi,$$

in which the fraction in the integrand changes sign only when $\phi-\theta=\Theta$ or $2\,\pi-\Theta$, the angle Θ being given by the formula

$$\Theta=\text{arc cos}\,\frac{2\,a\,r}{a^2+r^2}\cdot$$

If we denote by ρ the distance $M'\,M$, the corresponding value of ρ will be

$$\rho_0=\frac{a^2-r^2}{\sqrt{a^2+r^2}},$$

and if we denote by α the angle

$$\alpha=\arctan\frac{y-y'}{x-x'},$$

the corresponding values of α will be

$$\alpha_0=\pm\,\text{arc cos}\,\frac{r}{\sqrt{a^2+r^2}}\cdot$$

By means of (57), we have now

$$\frac{\partial u}{\partial r}=\frac{-1}{\pi}\int_0^{2\pi}f(M)\,\frac{\partial^2}{\partial r\,\partial\theta}\arctan\frac{y-y'}{x-x'}\,d\,\phi,$$

and,

$$\left|\frac{\partial u}{\partial r}\right|\leq\frac{N}{\pi}\int_{-\Theta}^{\Theta}+\frac{\partial^2}{\partial r\,\partial\phi}\arctan\frac{y}{x}\,d\,\phi+$$

$$\frac{N}{\pi}\int_{\Theta}^{2\pi-\Theta}-\frac{\partial^2}{\partial r\,\partial\phi}\arctan\frac{y}{x}\,d\,\phi$$

$$\leq\frac{4\,N}{\pi}\,\frac{\partial}{\partial r}\,\alpha_0$$

$$\leq\frac{4\,N}{\pi}\,\frac{\partial}{\partial r}\,\text{arc cos}\,\frac{r}{\sqrt{a^2+r^2}},$$

whence,

(58) $$\left|\frac{\partial u}{\partial r}\right|\leq\frac{4\,N}{\pi}\,\frac{a}{a^2+r^2},$$

and

$$N\,a\int_0^{\pi 2}\left|\frac{\partial u}{\partial r}\right|d\theta\leq 8\,N^2\frac{a^2}{a^2+r^2},$$

which approaches the limit

$$4\,N^2,$$

as r approaches a. The point is thus demonstrated.

13.42. It remains to show that the function given by (57) is really the function we seek, and takes on the boundary values $f(M)$ almost everywhere. To this end, we may restrict ourselves to a function $f(M)$ which is measurable (B), since such a function can be found which differs from our given function, summable (L), only on the points of a set of zero measure. Any function measurable (B), however, can be built up in a finite number of steps of the following kind, from the class of continuous functions:

(*a*) By taking limits of increasing sequences of functions measurable (B).

(*b*) By taking linear combinations of them. If $|f(M)|\leq N$, all these functions may be chosen of modulus $<N$.

It is known that if $f(M)$ is continuous, the harmonic function given in terms of it by Poisson's integral takes on the boundary values $f(M)$ continuously. Hence it will be sufficient to adopt a method of mathematical induction, and prove the following proposition.

13.43. Let $\{g_n(M)\}$ with $|g_n(M)|<N$ be a sequence of functions, measurable (B), given on the circumference of the circle, and increasing with n; and let $g(M)$ be the limit function. Let $u_n(M')$ be the function which is given in terms of $g_n(M)$ by means of Poisson's integral. Then if $u_n(M')$ takes on the boundary values $g_n(M)$ *almost everywhere*, the sequence $\{u_n(M')\}$ is a sequence of harmonic functions increasing with n, and the limit function $u(M)$ is

harmonic, is given in terms of $g(M)$ by Poisson's integral and takes on the boundary values $g(M)$ *almost everywhere.*

That $u_n(M')$ is harmonic and that the sequence is increasing follows obviously from the expression in terms of Poisson's integral. Therefore there is a limit function $u(M')$, and since $u_n(M')$ remains finite, irrespective of n, the integral equation remains satisfied in the limit:

$$u(M') = \frac{1}{2\pi}\int g(M) \frac{a^2 - r^2}{a^2 - 2ar\cos(\phi - \theta) + r^2} d\phi$$

and $u(M')$ is given in terms of $g(M)$ by Poisson's integral.

For simplicity now in investigating the boundary values of $u(M')$ let us take M_1 as the center 0 of the circle. If we take M_1 as any other point, the proof follows in a similar manner. With the notation M_r' for a point M' on the circumference of a circle of radius r, we have

$$\int_0^{2\pi} \{u(M_r') - u_n(M_r')\{ d\theta = 2\pi\{u(0) - u_n(0)\}$$

independently of r. The left-hand member can therefore be made as small as we please, independently of r by taking n large enough.

Take now n large enough so that at the same time,

$$\int_0^{2\pi} \{u(M_r') - u_n(M_r')\} d\theta \leq \frac{\epsilon}{3}$$

$$\int_0^{2\pi} \{g(M) - g_n(M)\} d\theta \leq \frac{\epsilon}{3},$$

and for this value of $\underline{n}$ take r large enough so that

$$\int_0^{2\pi} | g_n(M) - u_n(M_r') | d\theta \leq \frac{\epsilon}{3}$$

which we can do by hypothesis. We shall have then for this r,

$$\int_0^{2\pi} | g(M) - u(M_r') | d\theta \leq \epsilon;$$

whence,

$$\lim_{r=a}\int_0^{2\pi} | g(M) - u(M_r') | \, d\theta = 0. \tag{59}$$

But also, if we take any curve $r = \rho(\theta)$ continuous as a function of θ, $0 \leq \theta \leq 2\pi$ and such that $\bar{r} \leq \rho < a$, where $\bar{r}$ is the r of (59), we have

$$\int_0^{2\pi} | u(M_\rho') - u\, M_{\bar{r}}' | \, d\theta \leq \int_{\sigma_r} \frac{1}{r} \left| \frac{\partial u}{\partial r} \right| d\sigma,$$

where σ_r is the region contained between the two curves $r = \bar{r}$ and $r = \rho(\theta)$. From (58) it is seen that

$$\left| \frac{\partial u}{\partial r} \right| \leq \frac{4N}{\pi a}$$

so that this integral is $\leq C(a - \bar{r})$, where C is independent of $\bar{r}$. Hence from (59):

$$\lim_{\bar{r}=a}\int_0^{2\pi} | g(M) - u(M_\rho') | \, d\theta = 0.$$

At this point we can apply the lemma of Art. 13.31, modified in an obvious manner to fit the case of the circle, and deduce the equation

$$\lim_{r=a} u(M_r') = u(M)$$

as M' approaches M along the radius, for *almost all* values of θ. Thus the proof is complete.

14. Method of Poisson for summing a trigonometric series. Let $f(\theta)$ be any bounded function of θ, summable in the Lebesgue sense and with period 2π. Let a_k, b_k be its Fourier constants for $k = 0, 1, 2,$ and write the function

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_k^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta). \tag{60}$$

Poisson's methods for summing the series is to let r approach 1, and consider $\lim_{r=1} u(r, \theta)$

Summed according to the method of Poisson the series (60) *converges almost everywhere to the value* $f(\theta)$. In fact, when

$r < 1$ the series is convergent and its value is given by the Poisson integral in terms of $f(\theta)$. Hence, by the theorem of Art. 13,

$$\lim_{r=1} u(r, \theta) = f(\theta),$$

almost everywhere.

15. The extension to the finitely connected T-region. Since we have

$$\int_\sigma \left\{ \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right\} d\sigma = -\int_s g \frac{\partial g}{\partial n} ds,$$

g being the Green's function for T,

$$g(M_1, M) = \log \frac{1}{r} + g'(M_1, M),$$

and σ a region interior to T excluding the point M_1, with s for its complete boundary, if we take for σ the special region contained between two curves $g = k$, $g = k'$ and let k' approach zero we shall include in σ every point of T (since they are all interior points) except the points in the neighborhood of $g = \infty$.

We have

$$\int_\sigma \left\{ \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right\} d\sigma = (k - k') \, 2\pi, \tag{61}$$

and if we let k' approach zero

$$\int_\sigma \left\{ \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right\} d\sigma = 2k\pi.$$

Hence the squares of the partial derivative of $g(M_1, M)$ are summable over T when the neighborhood of M_1 is excluded.

If now the squares of the partial derivatives of $u(M)$ are summable over the region T, we have by Schwartz's inequality that the quantity

$$\frac{\partial g}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial u}{\partial y}$$

is summable over T, when the neighborhood of M_1 is excluded. Hence, by Green's theorem, we see precisely as in the case of the circle that the quantity

$$\lim_{g=0} \int_{h_1}^{h} u\, d\, h = H(h)$$

represents an absolutely continuous function of h, and with respect to the point M_1 defines a characteristic function $\bar{u}(h)$.

Now to every accessible point on the frontier of T corresponds a value of $h=c$, in that the point M' actually reaches the boundary point as it moves out along $h=c$; moreover to no two different accessible boundary points can correspond the same value $h=c$.[19]) It is easily shown that to *almost all* the values of h, $h=c$, correspond accessible points on the frontier of T.

In fact *almost all* of these curves $h=c$ are finite in length. Consider, in fact

$$\int_{g=1}^{0} d\, s_g = I_h \tag{62}$$

the length of a curve $h=c$ outside the point for which $g=1$, and form the integral

$$\int_0^{2\pi} I_h\, d\, h = \int_0^{2\pi} \left(\int_{g=1}^{0} d\, s_g \right) d\, h = \int_0^{1} d\, h \int^{0} \left| \frac{d\, s_g}{d\, g} \right| d\, g\, d\, h$$
$$= - \int \left| \frac{d\, s_g}{d\, g} \right| \frac{d\, g}{d\, s_g} \frac{d\, h}{d\, s_h}\, d\, \sigma$$

extended over the region T outside the curve $g=1$, provided this integral is convergent. But this integral is precisely

$$\int \left| \frac{d\, h}{d\, s_h} \right| d\, \sigma$$

which is

$$\int \left| \frac{d\, g}{d\, s_g} \right| d\, \sigma$$

and is convergent by (61). Hence the integral given by (62) has a finite value for almost all values of $h=c$.

[19]) Osgood and Taylor, Transactions of the American Mathematical Society, Vol. 14 (1913), pp. 277–298.

On the other hand, to a given accessible point there may correspond more than one value of h. It is known, however, that the closed curve formed by any pair of values of h corresponding to a single frontier point must contain other frontier points of T in its interior, and this frontier point may therefore be considered, as far as the geometric character of the region itself is concerned, as a point to be counted multiply.* With this proviso then, the function $\bar{u}(h)$ defines a function $f(M)$ on the accessible frontier points of T. With respect to $f(M)$ we have the following theorem.

15.1. *The function $f(M)$ is independent of M_1, the point chosen as pole. If $f(M)$ is given on the accessible frontier points of T, bounded, and for one pole M_1 summable with respect to h, there is one and only one bounded function $u(M)$ harmonic in T, with first derivatives summable with their squares over T, such that it takes on the boundary values $f(M)$ almost everywhere.*

If the property holds when one point M_1 is chosen as pole, the harmonic function is uniquely defined, and the property holds with respect to any pole.

To establish this theorem it is obviously sufficient to make use of the conformal transformation which transforms T into the circle, M_1 going into the center and $h=0$ into $\theta=0$. The only point to be proved is that a harmonic function the square of whose gradient is summable over T goes into a harmonic function the square of whose gradient is summable over the circle, and vice versa. But the integral of the square of the gradient is in fact an integral invariant whose value is given by the expression

$$\int\left\{\left(\frac{\partial u}{\partial g}\right)^2+\left(\frac{\partial u}{\partial h}\right)^2\right\} dg\, dh.$$

* See [19]).

16. The Stieltjes integral along the frontier. If we select a point M_1, and the Green's function $g(M_1, M)$, with its conjugate $h(M_1, M)$, it is an immediate consequence of the theorem of Green, that

$$(63) \qquad u(M_1) = \frac{1}{2\pi}\left\{\lim_{m=0}\int_{s_m} u(M)\, d\,h(M_1, M)\right\},$$

where s_m denotes the closed curve given by a constant value of the Green's function

$$g(M_1, M) = m, \qquad m \neq 0$$

if $f(M)$ is summable with respect to h and remains finite, this equation, with the aid of Art. 15, implies the result.

$$(64) \qquad u(M_1) = \frac{1}{2\pi}\int f(M)\, d\,h(M_1, M).$$

In what sense are we justified in saying that this is a Stieltjes integral of $f(M)$ extended over the boundary of the general open region T?

The accessible points of the boundary of T have intrinsically an order round the boundary. In fact, let A, B, C, D be four such points, and OA, OB, OC, OD be four curved rays from O to A, B, C, D not cutting each other. These curved rays from O have a circular order which may be generated through the relation of *betweenness*.

A conformal transformation shows that this order is independent of the point O. For if M_1 is the pole of a Green's function the point A will determine a value of $h(M_1, M)$ which is approached as M approaches A by any continuous curve such as the curve from the point O, — or, for instance, a curve $h(M_2, M) = \text{const.}$ belonging to another pole M_2. And a similar remark applies to B, C, D. Now if the region T is transformed conformally into a circle, the curves from a point O will be transformed into curves from a point O', interior to the circle, which cut each other only at

O', and the values of $h(M_1, M)$ corresponding to the points A, B, C, D will correspond to points A', B', C', D' on the circumference of the circle. And since the circular order of the curved rays from O' corresponding to A', B', C', D' is independent of the position of O', the same will be true of the curved rays from O; for the neighborhood of O is transformed conformally. Thus it may be said that the accessible points of the frontier of T possess an intrinsic order according to the relation of betweenness.

We are now in a position to answer the query with respect to equation (64). If the points of the boundary are taken in order, the Stieltjes sum for an integral like that in (64) is exactly the Riemann sum of the integral with respect to h itself; and consequently both have limiting values and determine integrals, which are equal, under the same condition. And if $f(M)$ is summable in the Borel sense with respect to the variable of integration h, the integral of (64) is defined by the theorems of Art. 1. 2 in terms of functions $f(M)$ for which the Stieltjes sum exists. We may therefore regard the boundary integral as properly a Stieltjes integral for functions defined on the accessible points of the boundary and measurable on this set of points in what may be called the Borel sense. A further extension to Lebesgue summability seems hardly desirable, according to the point of view of this paper.

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